§2. Modelling.

- Mathematic language to describe.
  Mathematic method to analyze.
  ⇒ mathematic model.

- Different components . controller ...

- Different signals.

- Connection of components/devices
  process of signal transferring

- Use mathematic, model to represent the dynamic process of signal flowing & transfering.

- Two ways
  - × Theoretical analysis: $F=ma$.
  - Experimental: regression.

- Dynamic. integral, derivative, ordinary differential equation.

$$a_2\ddot{y}+a_1\dot{y}+a_0\cdot y=b_0x$$

- Time invariant physical parameter x change along time

- Linear.

2.1 Linear System (a special and quite general type)

- $\boxed{\text{Superposition}}$: $\xrightarrow{x(t)}\boxed{G}\xrightarrow{y(t)}$

$$G(x_1(t) + x_2(t)) = G(x_1(t)) + G(x_2(t))$$
$$= y_1(t) + y_2(t) \qquad \forall \text{ inputs}$$

or. $\xrightarrow{x_1}\boxed{G}\xrightarrow{y_1}$, $\xrightarrow{x_2}\boxed{G}\xrightarrow{y_2}$ $\overset{\text{Linear}}{\Longrightarrow}$ $\xrightarrow{x_1+x_2}\boxed{G}\xrightarrow{y_1+y_2}$.

- $\boxed{\text{Homogeneity}}$

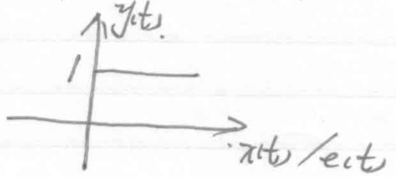
$$G(a x(t)) = a\, G(x(t))$$
$$= a\, y(t). \qquad \forall \text{ inputs}$$

$\xrightarrow{a x(t)}\boxed{G}\xrightarrow{a y(t)}$.

Are the following system linear / nonlinear.?

- $\underset{e(t)}{\xrightarrow{x(t)}}\boxed{G}\xrightarrow{y(t)}$

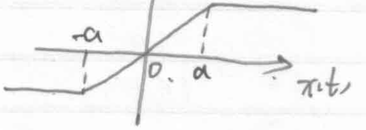
Thermostat



Violate both

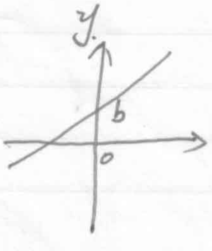
$$y(t) = \begin{cases} 1 & x(t) \geq 0 \\ 0 & x(t) < 0 \end{cases}$$

- Saturation: e.g.: application in flow control.



$$y(t) = \begin{cases} x(t) & -a < x(t) \leq a \\ 0 & \text{else.} \end{cases}$$

- Challenging



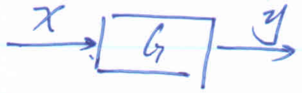
$y = ax + b$

$$a(kx) + b$$
$$= akx + b \neq ky$$

※ Note: No practical system is strictly linear.

- Some of the systems are inherently nonlinear. like 1. 2.
- Some of the systems can be treated as linear.

Define. $\bar{y} = y - b$.



$\underline{\underline{\bar{y} = ax}}$. linear.

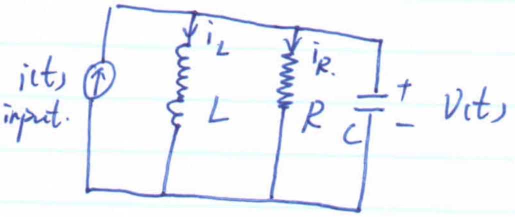
Why we study linear system?

- Simple.
- Basic for all systems.
- Beautiful characteristics.

✗ Many linear systems can be modelled by linear ordinary differential equations. (ODE)

Example: RLC circuit

capacitance.
resistance
inductance.



Kirchhoff's current law

$i_C + i_R + i_L = i$

$i_C(t) = C\frac{dv}{dt}$

$i_R(t) = V/R$.

~~$V(t) = L\dot{i}_L R$.~~

$V(t) = L\frac{di_L}{dt}$

$$\Longrightarrow C\dot{v}(t) + \frac{v(t)}{R} + \frac{1}{L}\int_{-\infty}^{t} v(t)dt = i(t)$$

$$\int_{-\infty}^{t} v(t)dt \triangleq x(t)$$

T_m: mechatronic time constant
T_a: loop electromagnetic constant

Input: u_a. M_L
control disturbance
Output: Ω.

If Output θ.

$$T_a T_m \frac{d^3\theta}{dt^3} + T_m \frac{d^2\theta}{dt^2} + \frac{d\theta}{dt} = \frac{1}{k_e} u_a - \frac{T_m}{J} M_L - \frac{T_a T_m}{J} \frac{dM_L}{dt}.$$

Linear Systems, Superposition.

$M_L = 0$ $$T_a T_m \frac{d^2\Omega_1}{dt^2} + T_m \frac{d\Omega_1}{dt} + \Omega_1 = \frac{1}{k_e} u_a$$

$u_a = 0.$ $$T_a T_m \frac{d^2\Omega_2}{dt^2} + T_m \frac{d\Omega_2}{dt} + \Omega_2 = -\frac{T_m}{J} M_L - \frac{T_a T_m}{J} \frac{dM_L}{dt}$$

$\Omega = \Omega_1 + \Omega_2.$

Ω_1: speed under u_a
Ω_2: speed under M_L.

§ 2.1 ODE

- Input & Output are ~~all~~ functions of time t.
- Most of the devices. Input, output can be represented by ODE.
- Input. output. & their derivatives are included in ODE.
- We call this ODE dynamic equations.
- Order of the ODE is the highest order of the derivatives, also the order of system
- Single input — Single output SISO.

$$y^{(n)}(t) + a_{n-1} y^{(n-1)}(t) + a_{n-2} y^{(n-2)}(t) + \cdots + a_0 y(t) =$$
$$b_m x^{(m)}(t) + b_{m-1} x^{(m-1)}(t) + \cdots + b_0 x(t)$$

$x(t)$ — Input
$y(t)$ — Output
$y^{(n)}(t)$ — n th order derivative of $y(t)$
a_i, b_j coefficients determined by system structure & components.

- Method:
  ① Analyze the principle of system, determine input & output of each component
  ② Write equations.
  ③ Eliminate inter variables. solve for ODE from system input to output. Only system IO & their derivatives included.

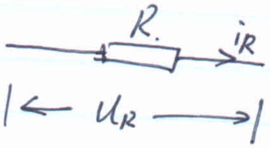
④ Simplify. Output & derivatives left
Input & derivatives right
In ~~order~~ deceding. order

2.1.1 Electrical network system

Most commonly used device.

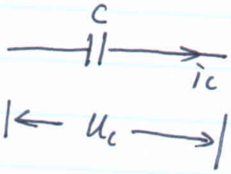
Resistance, ~~Conductance~~, capacitance inductance
→ no source.

~~Resistance~~
Resistor



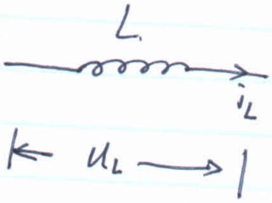
$u_R = R\, i_R$

~~Capacitance~~
Capacitor



$i_C = C \frac{du_c}{dt}$

~~Inductance~~
Inductor

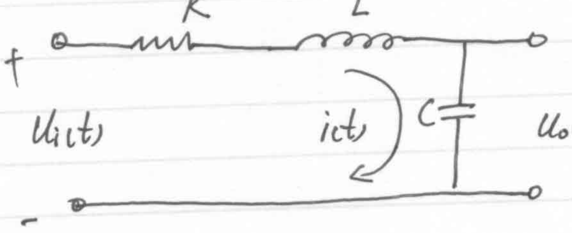


$u_L = L \frac{di_L}{dt}$

Kirchhoff's current law
voltage law

① Current flows in node = current flows out.
② Loop voltage sum = 0.

Example: passive devices.



$u_i(t)$ input Construct the ODE
$u_o(t)$ output

Solutions: $\Sigma u = 0$

$$u_L(t) + u_R(t) + u_c(t) = u_i(t)$$

$$L\frac{di(t)}{dt} + Ri(t) + u_o(t) = u_i(t)$$

$i(t)$ is inner variable, eliminate it

$$i(t) = C\frac{du_o(t)}{dt}$$

$$LC\frac{d^2u_o(t)}{dt^2} + RC\frac{du_o(t)}{dt} + u_o(t) = u_i(t)$$

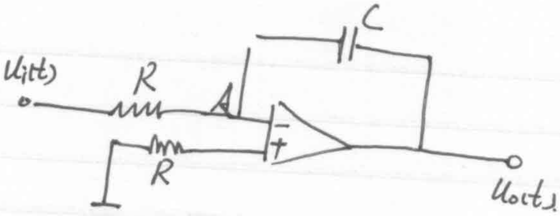
$$\Rightarrow T_1T_2\frac{d^2u_o(t)}{dt^2} + T_2\frac{du_o(t)}{dt} + u_o(t) = u_i(t)$$

$T_1 = L/R$ $T_2 = RC$

Typical 2nd-order ODE

Example: Operational Amplifier active device.

$u_i(t)$ input
$u_o(t)$ output



+ − . voltage same, current = 0.

$$A: \quad \frac{u_i(t)}{R} + C\frac{du_o(t)}{dt} = 0$$

$$\Rightarrow \quad RC\frac{du_o(t)}{dt} = -u_i(t)$$

$$\text{or} \quad T\frac{du_o(t)}{dt} = -u_i(t)$$

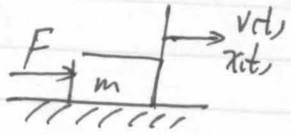
1st-order ODE

$T \rightarrow$ time constant $\quad T = RC$

2.1.2 Mechanical System

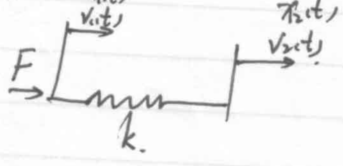
Three Basic elements: Mass, spring & damper.

mass:



$$F = m\frac{dv}{dt} = m\frac{d^2x}{dt^2}$$

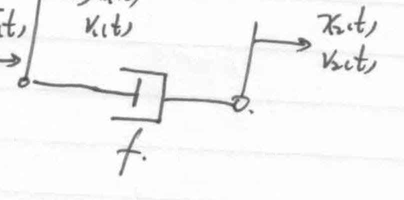
spring:



$$F = k(x_1 - x_2) = kx \qquad \text{stiffness } k$$

$$= k\int_0^t (v_1 - v_2)\,dt = k\int_0^t v\,dt$$

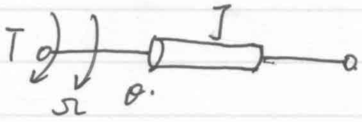
damper or damping:



$$F = f(v_1 - v_2) = fv$$

$$= f(\dot{x}_1 - \dot{x}_2) = f\dot{x}$$

Inertia:



$$T = J\frac{d\Omega}{dt} = J\frac{d^2\theta}{dt^2}$$

Newton's second Law:

Line. $\Sigma F = m\frac{d^2x}{dt^2}$

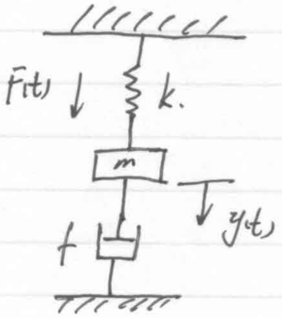
Rotation $\Sigma T = J\frac{d^2\theta}{dt^2}$

friction: Line: $F_c = F_B + F_f = f\frac{dx}{dt} + F_f$

$F_B = f\frac{dx}{dt}$ viscous friction

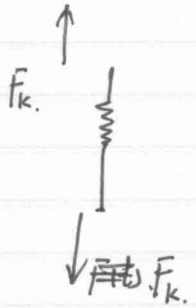
Rotation: $T_c = T_B + T_f = K_c\frac{d\theta}{dt} + T_f$

Example: Spring-mass-damper system



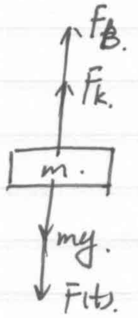
F(t) input
y(t) output

Free body diagram analysis.



$$F_k = k(y(t) + y_0)$$

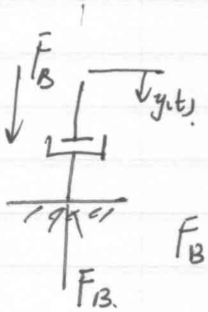
$$y_0 = mg/k$$



$$F(t) + mg - F_k - F_B = m\frac{d^2y(t)}{dt^2}$$

$$F_B = f\frac{dy(t)}{dt}$$

$$mg = ky_0$$



$$F_B = f\frac{dy(t)}{dt}$$

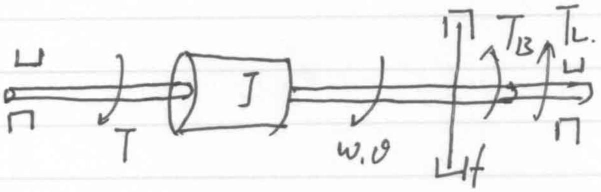
$$\Rightarrow \quad F(t) + ky_0 - ky(t) - ky_0 - f\frac{dy(t)}{dt} = m\frac{d^2y(t)}{dt^2}$$

$$\Rightarrow \quad m\frac{d^2y(t)}{dt^2} + f\frac{dy(t)}{dt} + ky(t) = F(t) \qquad \text{2nd-order ODE}$$

In the spring-mass-damper system, gravity doesn't appear in the final ODE.

Example: Mechanical rotation system.
Inertia load. damper.

T_L : Load torque.
T : Input torque.
Solve for $T \to \theta$ & $T \to \omega$.



Solution: $J \frac{d\omega}{dt} = T - T_B - T_L.$

$$T_B = f\omega. = f\frac{d\vartheta}{dt}$$

$$\Rightarrow J\frac{d\omega}{dt} + f\omega = T - T_L \qquad T_L: \text{disturbance.}$$

$$\Longleftrightarrow J\frac{d^2\vartheta}{dt^2} + f\frac{d\vartheta}{dt} = T - T_L.$$

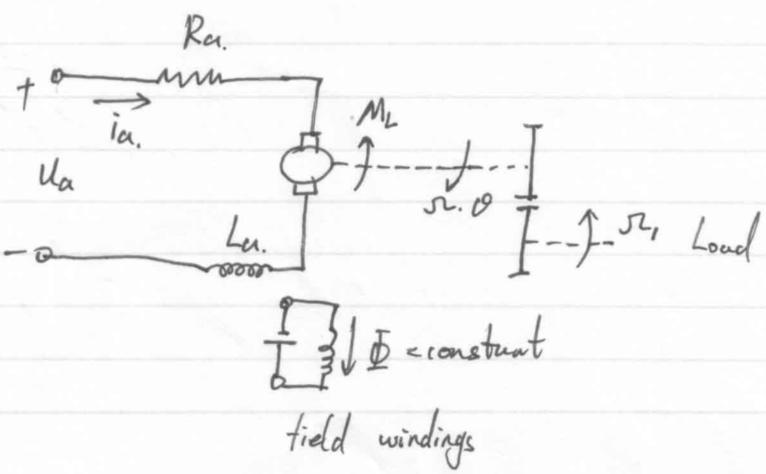
2.1.3. Mechatronics. System

- Mechanical & Electromagnetical components.
- Transformation of energy. electric energy $\rightleftarrows$ mechanical energy.

Example: Armature control DC motor.

Armature voltage u_a : Input.

Rotation speed Ω / ϑ : output.



$$L_a \frac{di_a}{dt} = \frac{L_a}{K_m} \frac{dM_D}{dt} = \frac{L_a}{K_m} \frac{d(J\frac{d\Omega}{dt} + M_L)}{dt} = \frac{L_a}{K_m} J \frac{d^2\Omega}{dt^2} + \frac{L_a}{K_m} \frac{dM_L}{dt}$$

$$R_a i_a = R_a \frac{M_D}{K_m} = R_a \frac{J}{K_m} \frac{d\Omega}{dt} + \frac{R_a}{K_m} M_L. \qquad e_a = k_e \Omega.$$

Solution:

(1) $L_a \frac{di_a}{dt} + R_a i_a + e_a = u_a$

$e_a = K_e \Omega.$

$$\Rightarrow L_a \frac{di_a}{dt} + R_a i_a + K_e \Omega = u_a \qquad ①$$

L_a: Inductance in the Loop H
R_a: Resistance in the loop Ω.
k_e: Back EMF V/rad s^{-1}
Ω: angular velocity rad/s
u_a: Control voltage applied on the amature V.
i_a: Amature current A
e_a: back. potential V

Motor axis $J\frac{d\Omega}{dt} + M_L = M_D$ ②

J: converted equivalent inertia. kg·m·s² J_L: Load inertia
M_L: converted equivalent load torque kg·m. J_m: Motor inertia
M_D: electromagnetic torque. kg·m.

Gear ratio $n = \frac{\Omega}{\Omega_1}$ $J = J_m + \frac{1}{n} J_L$

(2) M_D & i_a inner variables.

Flux Φ constant $\Rightarrow$ $M_D = K_m i_a$ ③ K_m: motor torque coef kg·m/A

(3) Eliminate inner variables

$$\frac{L_a J}{k_e K_m} \frac{d^2\Omega}{dt^2} + \frac{R_a J}{k_e K_m} \frac{d\Omega}{dt} + \Omega = \frac{1}{k_e} u_a - \frac{R_a}{K_e K_m} M_L - \frac{L_a}{K_e K_m} \frac{dM_L}{dt}$$

Let $\begin{cases} T_m = \frac{R_a J}{k_e K_m} \\ T_a = L_a / R_a. \end{cases}$ $\Rightarrow T_a T_m \frac{d^2\Omega}{dt^2} + T_m \frac{d\Omega}{dt} + \Omega = \frac{1}{k_e} u_a - \frac{T_m}{J} M_L - \frac{T_a T_m}{J} \frac{dM_L}{dt}$

Text book Section 2.2.

§ 3 Laplace Transform Review

§2 review : Linear System : Superposition.
Homogeneity.
Modelling : RLC
spring-mass

- In the end of last class, we mentioned a little bit of Laplace transform.

Motivation :
- Using transfer function instead of ODE to describe a system
  differential $\rightarrow$ algebraic.
  indirect $\longrightarrow$ direct
- Using frequency domain analysis instead of time domain

- Laplace transform comes from Fourier transform :

$\mathcal{F}$: Given $f(t)$ $\quad F(\omega) = \int_{-\infty}^{+\infty} f(t) e^{-2\pi i t\omega} dt$

we assume $f(t)$ is integrable

In signal processing

- Laplace transform. more realistic than Fourier, because in practical applications, we only care $f(t)$, $t \geq 0$

$\Rightarrow$ Assume: * $f(t) = 0$, $t < 0$.

Definition: $\underset{\substack{\downarrow \\ \text{function of complex variable } s.}}{\underline{F(s)}} = \mathcal{L}\{f(t)\} = \int_{0^-}^{\infty} f(t) e^{-st} dt$

* For the integral $F(s)$ to converge, $f(t)$ must be of exponential order as $t \to \infty$ ($f(t)$ cannot increase faster than exponential functions)

~~te^{st}~~ e^{t} X.

Exp① $f(t)=t$ $|e^{-st}f(t)|$ $=|e^{-\alpha t}t|$ $=\frac{t}{e^{\alpha t}}$ $\lim_{t\to\infty}\frac{t}{e^{\alpha t}}$ $=\lim_{t\to\infty}\frac{t'}{(e^{\alpha t})'}$ $=\lim_{t\to\infty}\frac{1}{\alpha e^{\alpha t}}$ $=\frac{1}{\infty}=0$.   Exp② $f(t)=$ ~~$e^{\alpha t}$~~ e^{e^t}

$\Leftrightarrow \int_{0^-}^{\infty}|f(t)e^{-st}|dt$

~~i.e., ∃ some β ∈ R such that $e^{\beta t}|f(t)| \to 0$ at $t\to\infty$~~

$|f(t)e^{-st}| \to 0$ as $t\to\infty$.

exponential order

$\Rightarrow$ ~~$e^{\alpha t}|f(t)|$~~ $\underline{|f(t)| \leq M e^{\alpha t}}$ $\exists M$. amplitude.

$\Rightarrow |f(t)e^{-st}| \leq |Me^{\alpha t}e^{-st}|$ exp: $f(t)=e^{2t}$

When analyzing the frequency characteristic,

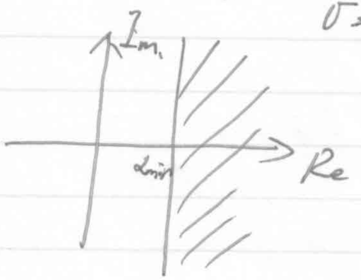
$s \to \sigma + j\omega$. σ.

$\sigma = \text{Re}\, s$. $\omega = \text{Im}\, s$

$|Me^{\alpha t}e^{-\sigma t}e^{-j\omega t}| = |Me^{(\alpha-\sigma)t}|$. for this to be integrable.

$\sigma > \alpha$. $\Rightarrow \text{Re}\, s > \alpha$.

In complex plane.

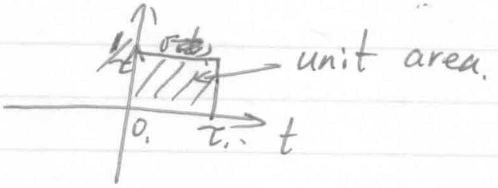


region of convergence.

- We're not going to talk more about the region because all of the signals in this course is integable for Laplace transform.

Ⅳ. Typical transform of signals.

① Unit impulse function.



$$f_\tau(t) = \begin{cases} 1/\tau & 0<t\leq\tau \\ 0 & \text{, else.} \end{cases}$$

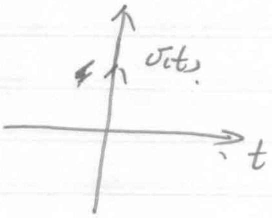
$F_\tau(s) = \int_{0^-}^{\infty} f_\tau(t)e^{-st}dt = \int_0^\tau \frac{1}{\tau}e^{-st}dt.$

$$= \begin{cases} 1 & s=0 \\ \frac{1}{\tau s}(1-e^{-s\tau}) & s\neq 0 \end{cases}$$

$\frac{1}{\tau}\frac{1}{(-s)}\int_0^\tau de^{-st}$

$= -\frac{1}{\tau s}e^{-st}\Big|_0^\tau = -\frac{1}{\tau s}(e^{-\tau s}-1) = \frac{1}{\tau s}(1-e^{-\tau s})$

② Unit impulse



$$\delta(t) = \lim_{\tau \to 0} f_\tau(t)$$

properties: $\delta(t) = 0, \quad t \neq 0$

$\delta(t) = \infty, \quad t = 0$

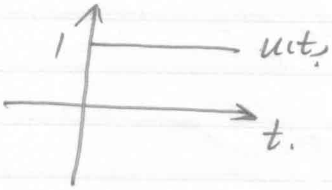
$$\int_{-\infty}^{\infty} \delta(t)dt = \int_{0^-}^{0^+} \delta(t)dt = 1$$

$$\mathcal{L}\{\delta(t)\} = \int_{0^-}^{\infty} \delta(t) e^{-st} dt$$
$$= \int_{0^-}^{0^+} \delta(t) e^{-s0} dt$$
$$= 1$$

Note: $\delta(t)$ is an idealized ~~signal~~ Disturbance signal. not practical, but is extremely important. (later)

③ Unit step

$$u(t) = \begin{cases} 1, & t \geq 0 \\ 0, & t < 0 \end{cases}$$



properties $\frac{du(t)}{dt} = \delta(t)$

$$\mathcal{L}\{u(t)\} = \int_{0^-}^{\infty} 1\, e^{-st} dt = -\frac{1}{s} e^{-st}\Big|_0^{\infty}$$
$$= -\frac{1}{s}(0-1)$$
$$= \frac{1}{s} \qquad s \neq 0$$

Properties of Laplace Transform.

⇒ ① Linearity $\quad \mathcal{L}(f_1) = F_1(s)$

$\mathcal{L}(f_2) = F_2(s)$

$$\mathcal{L}\{\alpha_1 f_1 + \alpha_2 f_2\} = \alpha_1 \mathcal{L}\{f_1\} + \alpha_2 \mathcal{L}\{f_2\}$$
$$= \alpha_1 F_1(s) + \alpha_2 F_2(s)$$

* ⇒ ② Differentiation

$$\mathcal{L}\{\dot f(t)\} = sF(s) - f(0^-)$$

: $\mathcal{L}\{\dot f(t)\} = \int_{0^-}^{\infty} \dot f(t)\, e^{-st}\, dt$ integration by parts

$$= \int_{0^-}^{\infty} e^{-st}\, d f(t)$$

$$= f(t) e^{-st}\Big|_{0^-}^{\infty} - \int_{0^-}^{\infty} f(t)(-s) e^{-st}\, dt$$

$$= 0 - f(0^-) + s\int_{0^-}^{\infty} f(t) e^{-st}\, dt$$

$$= sF(s) - f(0^-)$$

Conclusion: L.T. reduces <u>ode's differentiation</u> to <u>multiplication by s.</u>

Example: $\mathcal{L}\{\delta(t)\} = 1$

$\mathcal{L}\{u(t)\} = \frac{1}{s}$

$$\mathcal{L}(\dot u(t)) = s\cdot\frac{1}{s} - u(0^-) = 1 = \mathcal{L}\{\delta(t)\}$$

Extensions:

In 2nd-order ode's

$$\mathcal{L}\{\ddot f(t)\} = s\mathcal{L}\{\dot f(t)\} - \dot f(0^-)$$

$$= s(s\mathcal{L}\{f(t)\} - f(0^-)) - \dot f(0^-)$$

$$= s^2\mathcal{L}\{f(t)\} - sf(0^-) - \dot f(0^-)$$

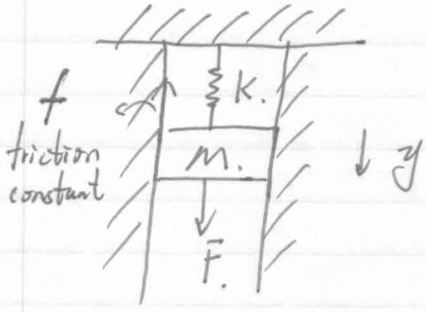
$$= s^2F(s) - sf(0^-) - \dot f(0^-)$$

$$\vdots$$

$$\mathcal{L}\{\frac{d^n}{dt^n} f(t)\} = s^nF(s) - s^{n-1}f(0^-) - \cdots - sf^{n-2}(0^-) - f^{n-1}(0^-)$$

$\Rightarrow$ $C\ddot{x} + \frac{1}{R}\dot{x} + \frac{1}{L}x = i$ $\Rightarrow$ 2nd-order linear ODE

Example: Spring - Mass Systems

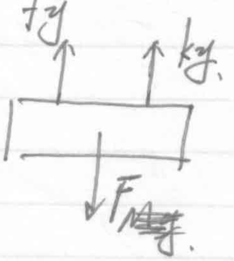


input: F force.

output: y displacement

To analyze this type of system, Free-body diagram.

- Assume: gravity has already been balanced by the initial stretch of the spring.



Newton's law: $\underline{\underline{F = m\ddot{x}}}$

$$M\ddot{y} = F - ky - f\dot{y}$$

$\Rightarrow$ $M\ddot{y} + f\dot{y} + ky = \underline{\underline{F(t)}}$ → 2nd-order linear ODE

- Compare the two system:

Electrical	Mechanical.
$i(t)$	$F(t)$
$\int_{-\infty}^{t} v(t)dt$	$y(t)$
$v(t)$	$\dot{y}(t)$
$R\ (v = Ri)$	$1/f\ (\dot{y} = \frac{1}{f}F)$
$L\ (v = Li)$	$1/k\ (\dot{y} = \frac{1}{k}\dot{F})$
$C\ (i = c\dot{v})$	$M\ (F = M\ddot{y})$.

Example: Mass-spring-damper.

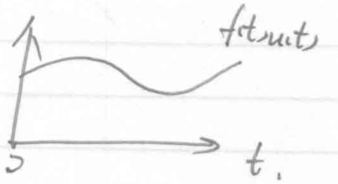
$$M\ddot{y} + f\dot{y} + ky = F(t)$$

Take L.T. with $y(0) = \dot{y}(0) = 0$ zero initial conditions

$$Ms^2Y(s) + fsY(s) + kY(s) = F(s)$$

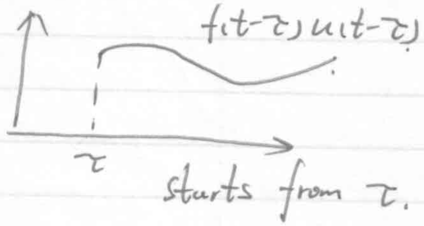
$$\Rightarrow Y(s) = \frac{1}{Ms^2+fs+k}\cdot F(s)$$

⇒. ③ Translation



In time domain $f(t) \longrightarrow f(t)u(t)$.

If there exists time delay τ.



Real:

$\mathcal{L}\{f(t-\tau)u(t-\tau)\}$

$= \int_{0^-}^{\infty} f(t-\tau)u(t-\tau)e^{-st}dt$ $t_1 = t-\tau \Rightarrow t = t_1+\tau$.

$= \int_{-\tau^-}^{\infty} f(t_1)u(t_1)e^{-\tau s}e^{-st_1}dt$

$= e^{-\tau s}\int_{(0^-)}^{\infty} f(t_1)u(t_1)e^{-st_1}dt_1$

$= e^{-\tau s}F(s)$

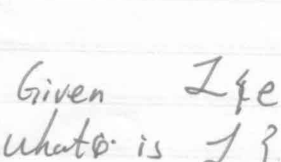
Question.: Given $\mathcal{L}\{e^t\} = \frac{1}{s-1}$

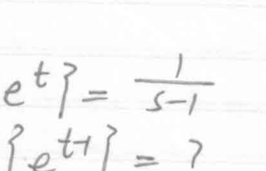
what is $\mathcal{L}\{e^{t-1}\} = ?$

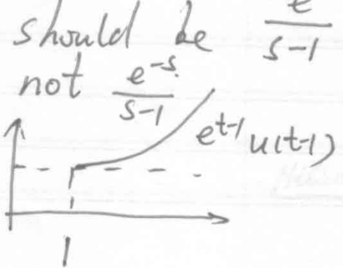
should be $\frac{e^{-1}}{s-1}$

not $\frac{e^{-s}}{s-1}$

because.







Complex: $\mathcal{L}\{e^{-at}f(t)\} = F(s+a)$

$$\int_{0^-}^{\infty} e^{-at} f(t)\, e^{-st}\, dt$$

$$= \int_{0^-}^{\infty} f(t)\, e^{-(s+a)t}\, dt$$

$$= F(s+a)$$

$$\mathcal{L}\{f(t)\} = \frac{s+2.}{s(s+1)^2(s^2+1.4s+4)}$$

example: $\mathcal{L}\{e^{-at}u(t)\} = \frac{1}{s+a}$ $\qquad \mathcal{L}\{e^{-2t}f(t)\}$

$\mathcal{L}\{e^{-at}t\,u(t)\} = \frac{1}{(s+a)^2}$

$$= \frac{s+2+2}{(s+2)(s+2+1)^2((s+2)^2+1.4(s+2)+4)}$$

Example: Find the L.T for $f_1(t) = \cos\omega t \quad f_2(t) = \sin\omega t,\ t \geq 0$

? Any idea

Apply Euler's Formula. $\quad e^{j\omega t} = \cos\omega t + j\sin\omega t$

$\qquad e^{-j\omega t} = \cos\omega t - j\sin\omega t.$

$$\text{L.T.} \begin{cases} \frac{1}{s-j\omega} = F_1(s) + jF_2(s) \\ \frac{1}{s+j\omega} = F_1(s) - jF_2(s) \end{cases}$$

$\Rightarrow F_1(s) = \frac{s}{s^2+\omega^2} \qquad F_2(s) = \frac{\omega}{s^2+\omega^2}$

In addition: $\mathcal{L}\{e^{-at}\cos\omega t\} = \frac{s+a}{(s+a)^2+\omega^2}$

$\mathcal{L}\{e^{-at}\sin\omega t\} = \frac{\omega.}{(s+a)^2+\omega^2}$

$\Rightarrow$ ④ Convolution (one of the reason of using Laplace)

block diagram

$$f_1(t) \rightarrow \boxed{f_2(t)} \rightarrow f_1(t)*f_2(t) \qquad F_1(s) \rightarrow \boxed{F_2(s)} \rightarrow F_1(s)\cdot F_2(s)$$

Suppose we have two functions $f_1(t)$ & $f_2(t)$, $t \geq 0$

$$f_1(t) * f_2(t) = \int_0^t f_1(t-\tau)\cdot f_2(\tau)\, d\tau$$

$$= \int_0^t f_2(t-\tau) f_1(\tau)\, d\tau$$

$$= f_2(t) * f_1(t)$$

$\mathcal{L}\{f_1(t) * f_2(t)\} = F_1(s)\cdot F_2(s)$ proof

-X- ⇒ Final Value Theorem. (can be used to analyze stability)

Assumption: $f(\infty) = \lim_{t\to\infty} f(t)$ exists.

OR $F(s)$ has all poles ~~at~~ in $\boxed{\mathrm{Re}\, s < 0}$ (LHP) except for $\boxed{\text{one}}$ possible pole at $s=0$.

Then $f(\infty) = \lim_{s\to 0} s\cdot F(s)$

3 Types $\overset{\text{poles}}{}$: 0. p (real). $\sigma \pm j\omega$ (complex)

In general, $F(s)$ can be written as

$$F(s) = \frac{A}{s} + \frac{B}{s-p} + \frac{Cs+D}{(s-\sigma)^2+\omega^2} + \cdots$$

$$\Downarrow$$

$$f(t) = A + Be^{pt} + e^{\sigma t}[\bar{C}\sin\omega t + \bar{D}\cos\omega t] + \cdots$$

as long as $p<0$ and $\sigma<0$. $\underline{f(t) \text{ can converge.}}$

$f(\infty)$ exist.

$$f(\infty) = A = \lim_{s\to 0} s\cdot F(s).$$

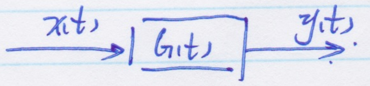
$$F(s) = \frac{5}{s(s+3)(s+4)}$$

$$f(\infty) = \lim_{s\to 0} s\cdot \frac{5}{s(s+3)(s+4)} = \frac{5}{7}$$

§4 Inverse Laplace Transform.

§3 review: Characteristics of Laplace Transform.

Advantage of L.T. ⇒ Simple.



Time domain: $y(t) = \int_0^t x(\tau)\, G(t-\tau)\, d\tau$. Integral → Multiplication

S domain: $Y(s) = X(s) \cdot G(s)$. Operation is straightforward but result is not.

How to get y(t) back?

$$Y(s) \xrightarrow{\mathcal{L}^{-1}} y(t). \quad \text{Definition } \frac{1}{2\pi i} \lim_{T\to\infty} \int_{r-iT}^{r+iT} e^{st} F(s)\, ds$$

Definition is not handy. the most frequently way is using the table. have to remember.

$\delta(t)$	$1.$
$u(t)$	$1/s.$
$t\,u(t)$	$1/s^2$
$t^n u(t)$	$n!/s^{n+1}$
$u(t-\tau)$	$e^{-\tau s}/s.$
$e^{-at} u(t)$	$\frac{1}{s+a}$
$\sin\omega t\, u(t)$	$\frac{\omega}{s^2+\omega^2}$
$\cos\omega t\, u(t)$	$\frac{s.}{s^2+\omega^2}$

Having this table, we can do inverse L.T. by using partial fraction expansion.

Example 1: Assume F(s) is real-rational

$F(s) = \frac{B(s)}{A(s)}$, A & B are polynomials.

<u>poles</u> of F are roots of A.
<u>zeros</u> of F are roots of B.

case 1 : (a) F(s) has distinct poles

$$F(s) = \frac{B(s)}{A(s)} = \frac{K(s+z_1)(s+z_2)\cdots(s+z_m)}{(s+p_1)(s+p_2)\cdots(s+p_n)} \qquad n>m.$$

$$\xrightarrow{PFE} \frac{a_1}{s+p_1} + \frac{a_2}{s+p_2} + \cdots + \frac{a_n}{s+p_n}$$

with $a_i = (s+p_i)\cdot F(s)\big|_{s=-p_i}$

$$\xrightarrow{Table.} f(t) = a_1 e^{-p_1 t} + a_2 e^{-p_2 t} + \cdots + a_n e^{-p_n t} \qquad \boxed{t\geq 0} \;*$$

$$\frac{1}{s^2+7s+12}$$

$$= \frac{1}{(s+3)(s+4)}$$

$$= \frac{a}{s+3} + \frac{b}{s+4}$$

$$= \frac{1}{s+3} - \frac{1}{s+4}$$

$$\downarrow$$

$$e^{-3t} - e^{-4t} \quad t\geq 0$$

Examples given in turtorial. ←

<u>from example</u> ← (b) $F(s) = \frac{2s+12}{s^2+2s+5}$

$$= \frac{2s+12}{(s+1+2j)(s+1-2j)} \qquad \text{Complex roots ?}$$

Use L.T. of sin cos.

$$F(s) = \frac{2(s+1)+10}{(s+1)^2+4}$$

$$f(t) = 2e^{-t}\cos 2t + \frac{10}{\underline{\underline{10}}} e^{-t}\sin 2t \qquad \boxed{t\geq 0} \;*$$

Case 2: Multiple poles.

Example: $F(s) = \frac{4}{(s+1)^2(s-1)} \xrightarrow{PFE} \frac{a_1}{(s+1)^2} + \frac{a_2}{s+1} + \frac{a_3}{s-1}$

$a_1 \quad a_2 \quad a_3$?

$$F(s) = \frac{a_1(s-1) + a_2(s+1)(s-1) + a_3(s+1)^2}{(s+1)^2(s-1)}$$

$$= \frac{a_1 s - a_1 + a_2 s^2 - a_2 + a_3 s^2 + 2a_3 s + a_3}{(s+1)^2(s-1)} = \frac{4}{(s+1)^2(s-1)}$$

$$\Rightarrow \begin{cases} a_2 + a_3 = 0. \\ a_1 + 2a_3 = 0. \\ -a_1 - a_2 + a_3 = 4 \end{cases} \Rightarrow \begin{cases} a_1 = -2. \\ a_2 = -1 \\ a_3 = 1 \end{cases}$$

$\Rightarrow f(t) = -2te^{-t} - e^{-t} + e^{t} \quad \boxed{t \geq 0}$

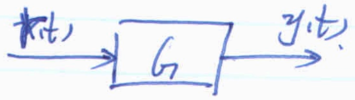
OR

$f(t) = (-2te^{-t} - e^{-t} + e^{t})\, u(t)$

§5 Transfer Function

§4 Review : Inverse Laplace Transform.
Using table

- Mentioned before.
- Systems (mass, spring damper) model can be described by ODE. (time domine)
- It describes the relationship between input & output.
- Solve 1st order ODE, 2nd easy. higher order hard.
- Using L.T. ODE ⟶ algebrac eqn.
- The model (ODE) in S domain is T.F.



ODE: $\frac{d^n y}{dt^n} + q_{n-1}\frac{d^{n-1} y}{dt^{n-1}} + \dots + q_0 y = p_{n-1}\frac{d^{n-1} r}{dt^{n-1}} + \dots + p_0 r.$

Assume: zero initial conditions

L.T.: $(s^n + q_{n-1}s^{n-1} + \dots + q_0)Y(s) = (p_{n-1}s^{n-1} + \dots + p_0)R(s)$

$Y(s) = G(s)R(s)$

$\Rightarrow G(s) = \frac{Y(s)}{R(s)} = \frac{p_{n-1}s^{n-1} + p_{n-2}s^{n-2} + \dots + p_0}{s^n + q_{n-1}s^{n-1} + \dots + q_0} = \frac{M(s)}{N(s)}$

$\Downarrow$

Transfer function

$\Rightarrow$ $Y(s) = G(s)\,R(s).$

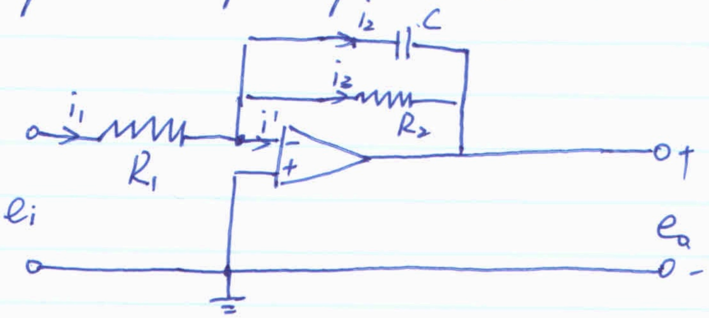
Compare s domain and time domain

$r(t) \xrightarrow{} \boxed{G.} \xrightarrow{} y(t)$ $\qquad y(t) = r(t) * g(t)$

$R(s) \xrightarrow{} \boxed{G(s)} \xrightarrow{} Y(s).$ $\qquad Y(s) = R(s)\,G(s).$

~~Ata~~

Example: Op-Amp Circuit.



TF: $e_i \to e_o$

Properties: ① High input impedence. $i' = 0.$

② High gain $e' = 0$ virtual ground.

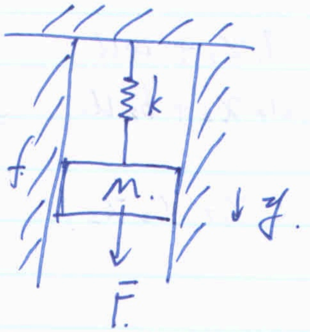
$\Rightarrow$ $i_1 = i_2 + i_3$

$$\Rightarrow \frac{e_i}{R_1} = -C\dot{e}_o - \frac{e_o}{R_2}.$$

$$E_i(s)/R_1 = -Cs\,E_o(s) - E_o(s)/R_2.$$

$$\Rightarrow \frac{E_o(s)}{E_i(s)} = \frac{-1/R_1}{Cs + 1/R_2} = \frac{-R_2/R_1}{R_2 C s + 1}$$

Example: Spring mass



TF : $F \rightarrow y$.

$$M\ddot{y} = F - ky - f\dot{y}$$

$$M s^2 Y(s) = F(s) - kY(s) - fsY(s)$$

$$\frac{Y(s)}{F(s)} = \frac{1}{Ms^2+fs+k}..$$

Properties of TF:

- TF is in s domain. It corresponds ODE in time domain.
- TF is an inherent character of the system, which is determined by ~~physical~~ system structer & components. No relationship with input signal.
- But, is related to the place where the input is injected to the system.

Op-Amp: $E_i \rightarrow E_o = \dfrac{-R_2/R_1}{R_2Cs+1}$

$I_1 \rightarrow E_o = \dfrac{-R_2}{R_2Cs+1}$

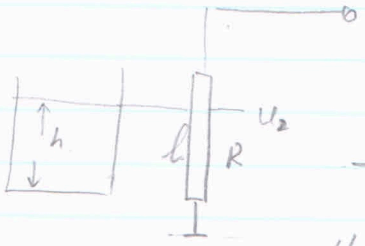
- In practical application. TF is a real-rational function of s. order of denominator > order of numeritor
- $N(s)=0 \triangleq$ characteristic equation

roots $\triangleq$ characteristic roots $\triangleq$ poles of ~~system~~ TF

- **Element:** Whole system is complicated
  Split into elements

Basic elements:

① Gain (Proportional)



$$\frac{U_2}{U_1} = \frac{h}{l}$$

$$U_2 = \frac{U_1}{l} h$$

ODE: $y(t) = K r(t)$

TF: $G(s) = \frac{Y(s)}{R(s)} = K$

Gear reducer, Potentiometer, Rotary Transformer.

② Inertia

ODE: $T \frac{dy(t)}{dt} + y(t) = r(t)$

TF: $G(s) = \frac{Y(s)}{R(s)} = \frac{1}{Ts+1}$

T: time constant for Inertia

③ Integral when $r(t)$ becomes 0. $y(t)$ will keep the value at the time $r(t)=0$

ODE: $y(t) = \int r(t)\, dt$

TF: $G(s) = \frac{Y(s)}{R(s)} = \frac{1}{s}$.

why $\frac{1}{s}$ integral

$\frac{1}{s} \to u(t)$

$\int_0^t r(\tau) u(t-\tau) d\tau$

$= \int_0^t r(\tau) d\tau$

④ Oscillation

ODE: $T^2 \frac{d^2 y(t)}{dt} + 2\zeta T \frac{dy(t)}{dt} + y(t) = r(t)$ $0<\zeta<1$

TF: $G(s) = \frac{Y(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$

T: time constant

ω_n: angular frequency $\omega_n = 1/T$

ζ: damping ratio

⑤ Pure derivative

ODE: $y(t) = \frac{dr(t)}{dt}$

TF: $G(s) = \frac{Y(s)}{R(s)} = s$

⑥ 1st-order derivative

ODE: $y(t) = \tau \frac{dr(t)}{dt} + r(t)$

TF: $G(s) = Y(s)/R(s) = \tau s + 1$ τ: time constant

⑦ 2nd-order derivative

ODE: $y(t) = \tau^2 \frac{d^2 r(t)}{dt^2} + 2\zeta\tau \frac{dr(t)}{dt} + r(t)$

TF: $G(s) = Y(s)/R(s) = \tau^2 s^2 + 2\zeta\tau s + 1$

⑧ Delay.

ODE: $y(t) = r(t-\tau)$

TF: $G(s) = Y(s)/R(s) = e^{-\tau s}$

Example: $T(s) = \frac{10(0,5s+1)}{s(s+1)(0,01s+1)}$

1 Proportional: $K=10$.
1 1st-order deri: $0.5s+1$
1 Integral: $1/s$
2 Inertia: $1/s+1$ $1/0.01s+1$

① pole-zero

② Real rational

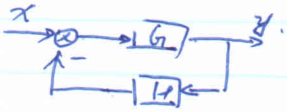
$\frac{500s+1000}{s^3+101s^2+100s}$

§ 6 Block Diagram. (Section 5.2)

§ 5 Review: Transfer function
Basic elements.

- Use a block to represent for a basic element, or a complicated transfer function

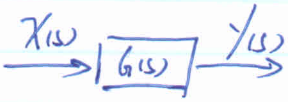
$\frac{G}{1+GH}$



- Can be used to calculate T.F.

Block Diagram: Input, Output, T.F.

signal flow
transfer

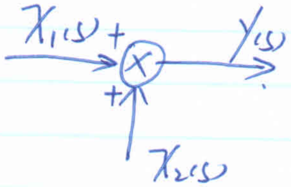


$Y(s) = X(s)\,G(s)$ Basic.

---

① $X(s)$. $Y(s)$ ⟹ Signal.

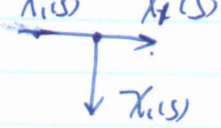
② $\otimes$ ⟹ Summing point

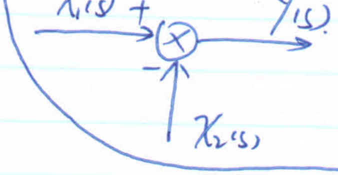


※. Indicate "+" "−"

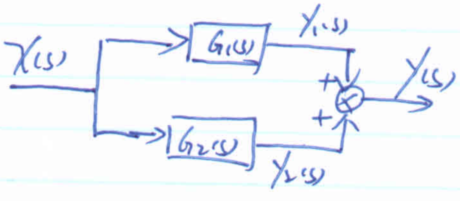
③  ⟹ Branch Point/Pickoff point

not like KCL

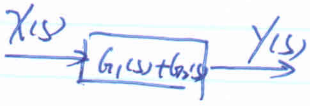




- Complicated diagrams ⟶ Basic one, using some reduction techniques

① 

⟺



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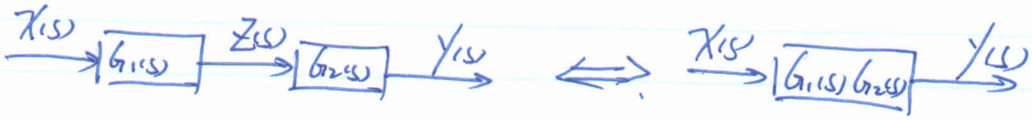

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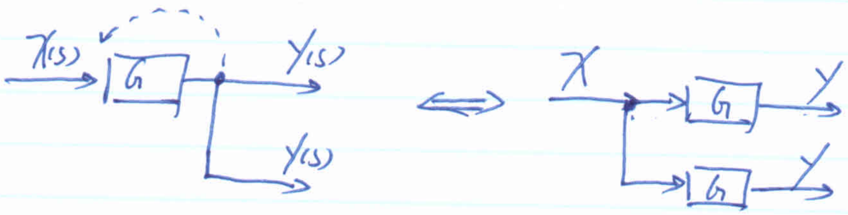
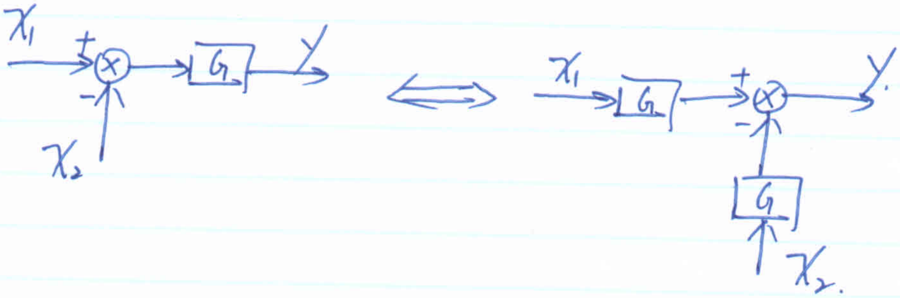


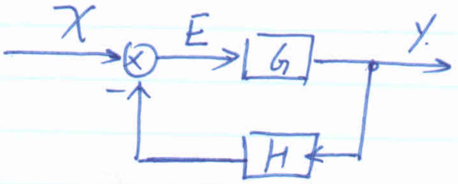
Moving branch point.

④



Moving summing point.

⑤ Feedback (negative)



$$Y = G(X - HY)$$

$$Y = \frac{G}{1+GH} X$$

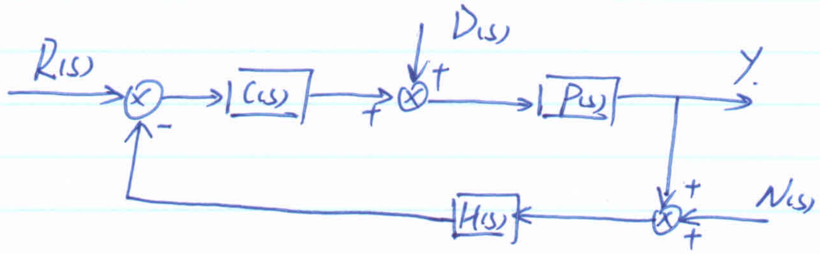
G: Open-loop T.F.

$$\frac{G}{1+GH} = \frac{\text{Open-loop T.F}}{1 + \underbrace{\text{Open-loop T.F} \times \text{F.B loop T.F}}_{\text{"loop gain"}}} = \text{closed-loop T.F}$$

Positive Feedback. "–"

Example: Single-loop Control System

explain:



DC motor: $R(s)$: Speed.

$C(s)$: Control chip

$D(s)$: Disturbance

$P(s)$: Motor

$N(s)$: Noise for sensor.

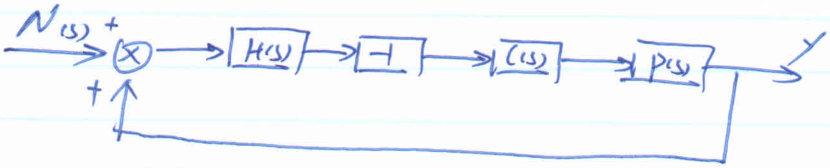
$H(s)$: Sensor

Three inputs: (Linear System, superposition)

$R \to Y$: $\dfrac{PC}{1+PCH}$ $D=0=N.$

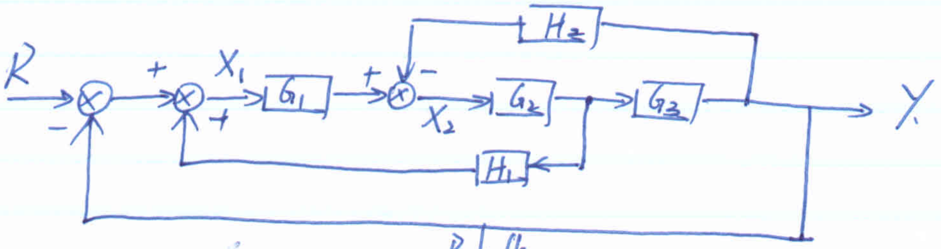
$D \to Y$: $\dfrac{P}{1+PCH}$ $R=0=N.$

$N \to Y$: $\dfrac{-PCH}{1+PCH}$ $R=0=D$

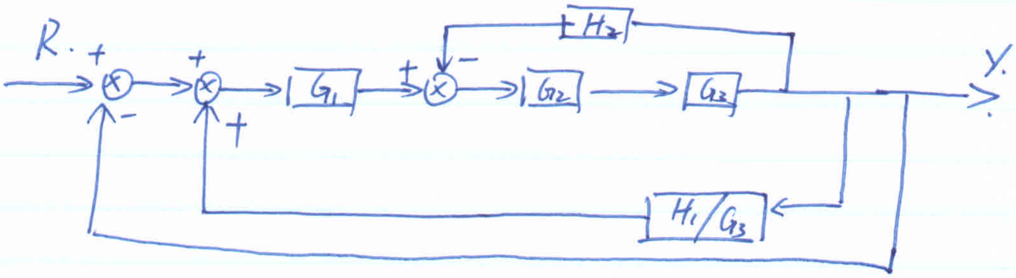


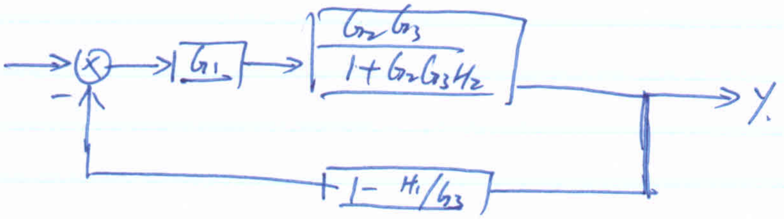
R.D.N : $Y = \dfrac{PC}{1+PCH} R + \dfrac{P}{1+PCH} D + \dfrac{-PCH}{1+PCH} N.$

Example: Multi-loop Control System
(Constructing basic loop)



※ : Summing point & Pickoff/Branch point cannot exchange positions.





$$\Rightarrow \quad Y/R = \frac{\dfrac{G_1 G_2 G_3}{1+G_2G_3H_2}}{1+\dfrac{G_1G_2G_3}{1+G_2G_3H_2}(1-H_1/G_3)}$$

Algebraic Method:

① Label the outputs of the summing point.

② Label the inputs of the summing point

③ Write the equations

$$\begin{cases} X_1 = R - Y + H_1 G_2 X_2 \\ X_2 = G_1 X_1 - H_2 Y. \\ Y = G_3 G_2 X_2 \end{cases}$$

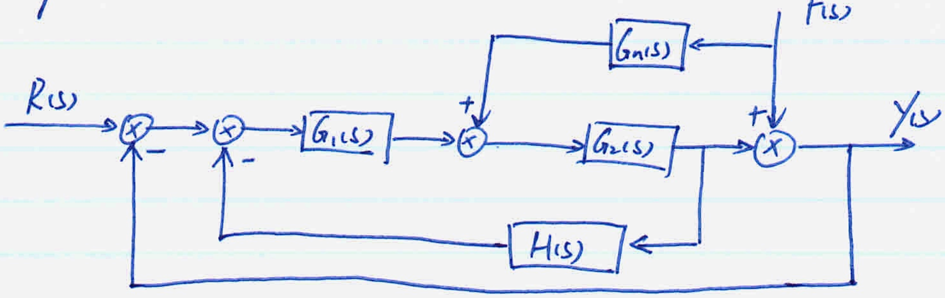
④ Solve for Y with R known.

$$Y = \frac{G_1 G_2 G_3}{1 + G_2 G_3 H_2 + G_1 G_2 G_3 - G_1 G_2 H_1}$$

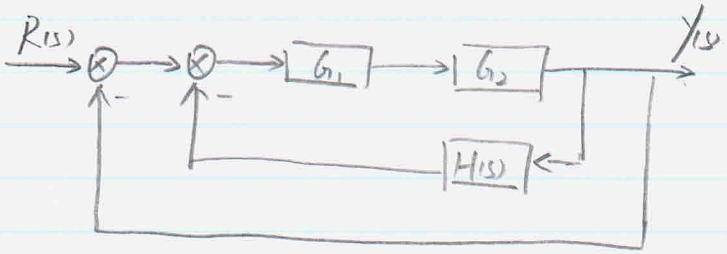
Signal Flow & Mason's formula.

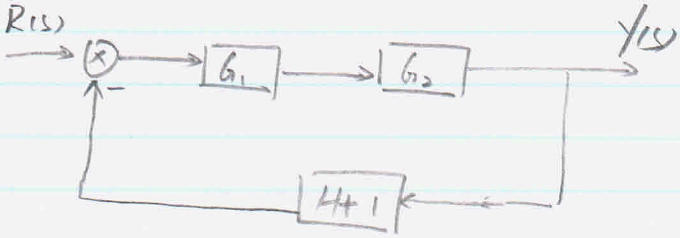
too complicated, in Turtorial.

Example:



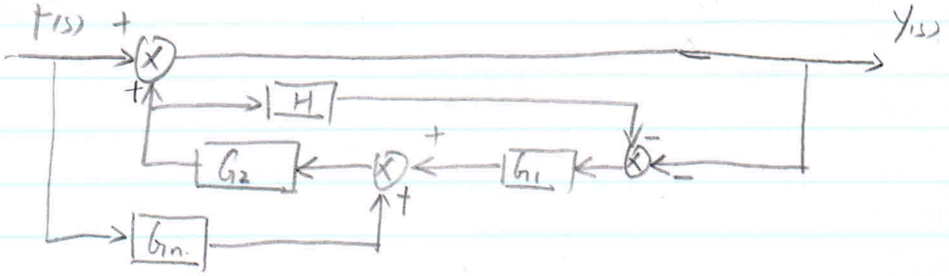
Solution: $R(s) \rightarrow Y(s)$

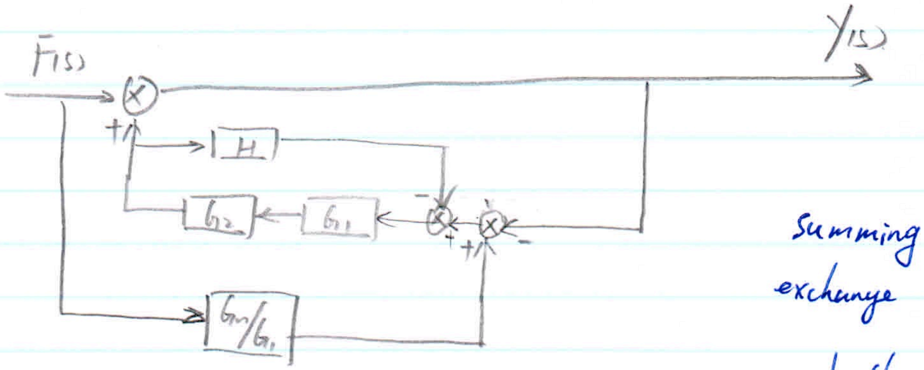




$$\frac{Y(s)}{R(s)} = \frac{G_1 G_2}{1+ G_1 G_2 (H+1)}$$

$F(s) \rightarrow Y(s)$

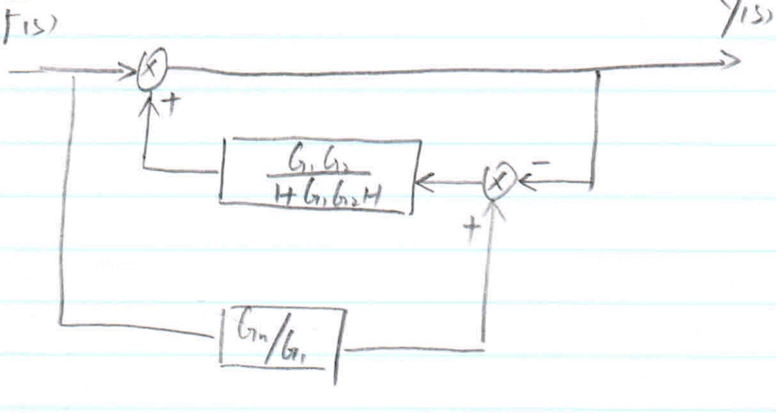


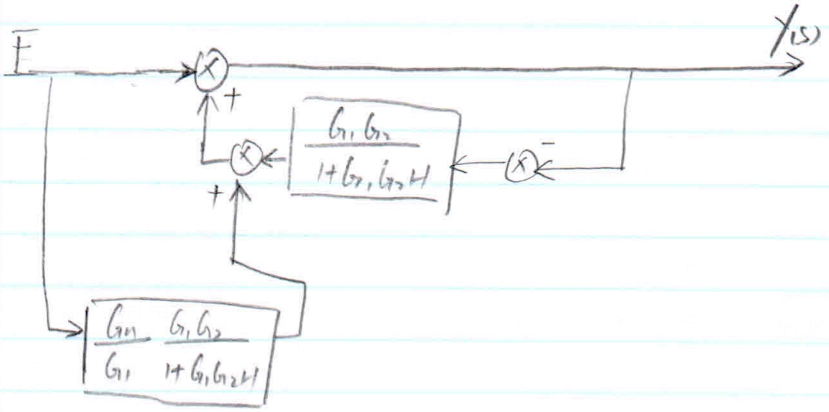


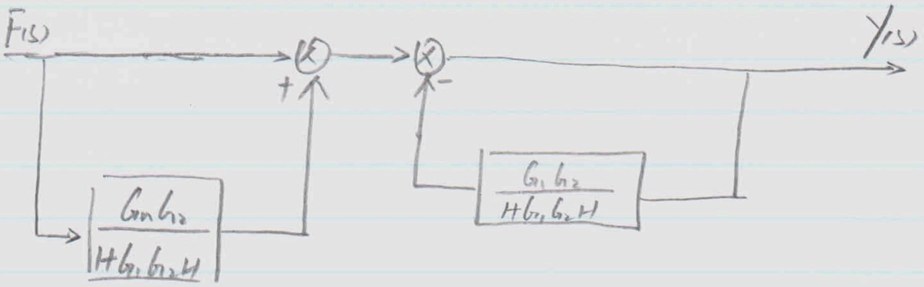
Summing points can. exchange their positions

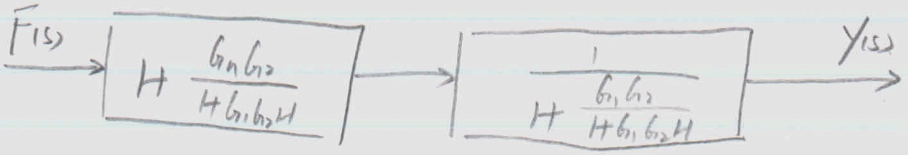
pickoff points can — —

Summing & Pickoff cannot









$$\frac{1+G_{11}G_{12}H+G_{11}G_{12}}{1+G_{11}G_{12}H} \qquad \frac{1+G_{11}G_{12}H}{1+G_{11}G_{12}H+G_{11}G_{12}}$$

$$= \frac{1+G_{11}G_{12}H+G_{11}G_{12}}{1+G_{11}G_{12}+G_{11}G_{12}H}$$

§7 Time Domain Analysis & Synthesis

§6 Review. Block Diagram Reduction

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§7.1 Typical Input Signal

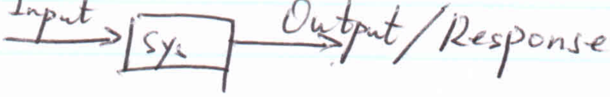
Design Control } Analysis : Analyze the performance
Synthesis : Improve the performance.

The Last several class: Constructing T.F.

- Method used in this chapter:

① Given. closed-loop T.F $\Phi(s)$
② Using $Y(s) = \Phi(s) R(s)$ get $Y(s)$. I.L.T get $y(t)$
③ Using $y(t)$ to analyze stability, dynamic quality & steady state error
and. Justify if the system satisfy the requirement
④ Using $Y(s)$ to analyze
⑤ Adjust the parameter to improve the performance.

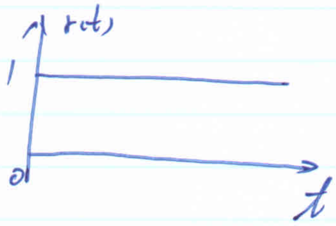
- Typical input signal:



① Can reflect the system performance in a specific view
like Quick, ~~steady~~ stable, Accurate, steady
② Function simple.
③ L.T simple.

⟹ Unit step

$$r(t) = \begin{cases} 1 & t \geq 0 \\ 0 & t < 0 \end{cases}$$



$$R(s) = \frac{1}{s}.$$

By using Unit step, we require the system output change its value rapidly. So it can reflect $\begin{cases} \text{Quick.} \\ \text{steady} \end{cases}$

⟹ Ramp.

$$r(t) = \begin{cases} t & t \geq 0 \\ 0 & t < 0 \end{cases}$$

$$R(s) = \frac{1}{s^2}$$

Performance of tracking an uniform speed signal.

⟹ Acceleration

$$r(t) = \begin{cases} \frac{1}{2}t^2 & t \geq 0 \\ 0 & t < 0 \end{cases}$$

$$R(s) = \frac{1}{s^3}$$

1/2 to make R(s) simple.

⇒ Unit Impulse

$$\begin{cases} r(t) = \delta(t) = \begin{cases} \infty & t=0 \\ 0 & t\neq 0 \end{cases} \\ \int_{-\infty}^{+\infty} \delta(t)\,dt = 1 \end{cases}$$

$R(s) = 1$

~~Pef~~ Performance. of being disturbed or impulse applied.

⇒ Sinusoidal

$r(t) = A \sin\omega t$ $\qquad$ $r(t) = A \cos\omega t$

$R(s) = A \frac{\omega}{s^2+w^2}$ $\qquad$ $R(s) = A \frac{s}{s^2+w^2}$

A: amplitude.
w: angular frequency.

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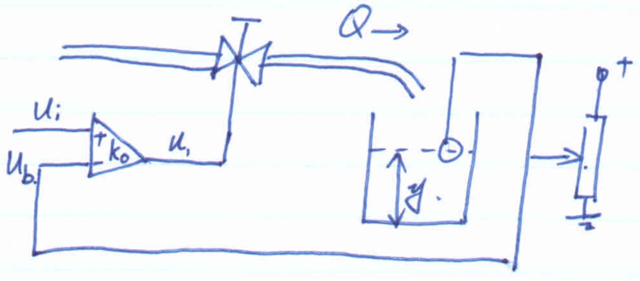
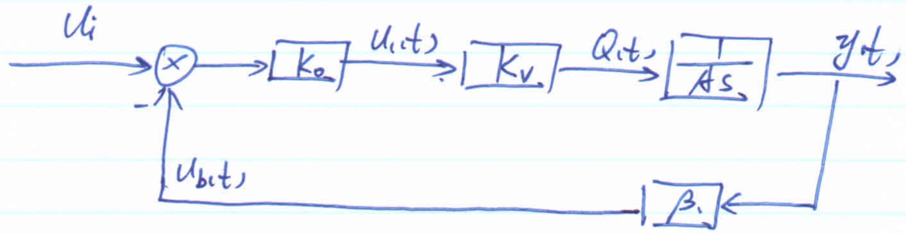
section 4.3

§ 7.2 First-Order System (1st-order system)

Simplist one: Liquid Level Control System

• Model:





Amplifier: $U_1 = K_o(U_i - U_b)$ Q : flow rate L/s.

Valve: $Q = K_v U_1$

Tank: $y = \frac{1}{A}\int Q\,dt.$

Feedback potentiometer: $U_b = \beta \cdot y.$

$$\Phi(s) = \frac{Y(s)}{U_i(s)} = \frac{K_o K_v \frac{1}{As}}{1 + k_o K_v \frac{1}{As}\beta} = \frac{1/\beta}{\frac{A}{k_o K_v \beta}s+1}$$

$$= \frac{1}{Ts+1}\cdot\frac{1}{\beta}$$

$$T = \frac{A}{k_o K_v \beta}.$$

General: $\Phi(s) = \frac{1}{TS+1}$

*If any gain K exists in the system, according to the linear system theory, $\Phi(s) = \frac{K}{TS+1}$.

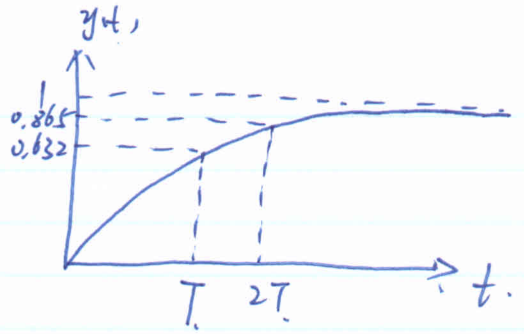
- Step Response

$$Y(s) = \Phi(s)\cdot R(s) \qquad y(t) = 1 - e^{-t/T} \quad t \geq 0.$$

$$= \frac{1}{TS+1}\cdot\frac{1}{S}$$

$$= \frac{1}{S} - \frac{T}{TS+1}$$

$$= \frac{1}{S} - \frac{1}{S+\frac{1}{T}}$$



① y(t) Initial = 0.
Final = 1.
Monotonically rising

② y(t) rising according to exponential function

t = T	y(t) = 63.2%
2T	86.5%
3T	95%
4T	98.2%
5T	99.3%

$t \to \infty$. $y(t) \to 1$, but if y(t) is within a specific range, we can concider the dynamic process ends

Quick → Reduce "T"

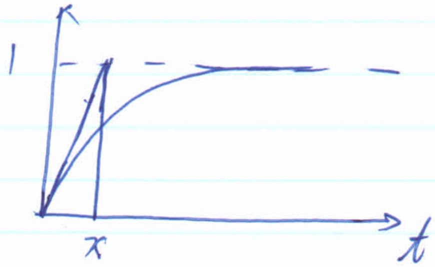
③ For unit negative feedback. $T > 0$ stable
otherwise $1 - e^{-t/T}$ go crazy

④ $\frac{1}{TS+1}$ T.F. only one pole at $S = -\frac{1}{T}$. as long as. P at LHS → stable,

pole far away from Image axis → Quick.
T↓.

⑤ By experiment, we can easily obtain the parameter T.

$t=T$ $y=63.2\%$ From data



$$\frac{dy}{dt} = \frac{1}{T}e^{-t/T}$$

$$\left.\frac{dy}{dt}\right|_{t=0} = \frac{1}{T}$$

at $t=0$. slope is $\frac{1}{T}$. $\frac{1}{x}=\frac{1}{T}$ $\underline{\underline{x=T}}$ From curve

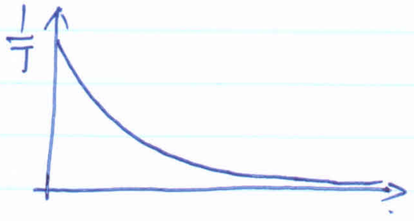
$r(t)=K\,u(t)$ $y_{ss}=K$.

- Unit Impulse Response

Input: Unit Impulse L.T 1.
Output: T.F

$$Y(s)=\Phi(s)R(s)=\frac{1}{Ts+1}$$

$$y(t)=\mathcal{L}^{-1}\left\{\frac{1}{Ts+1}\right\}=\frac{1}{T}e^{-t/T} \quad t\geq 0$$



Monotonically decreasing

$t=$	0	T	2T	3T	4T	∞.
	$\frac{1}{T}$	$0.368\frac{1}{T}$	$0.135\frac{1}{T}$	$0.05\frac{1}{T}$	$0.018\frac{1}{T}$	0

If we define the settling time t_s at which $y(t) \leq 5\%\frac{1}{T}$. / 2%

Then $t_s = 3T$ / $4T$. T reflects quickness

✗ : • With 0 initial condition. unit impulse response is the T.F.

- We always use IR to get the closed-loop transfer function
- In practical, impossible to get ideal impulse signal. Instead of that, using a rectangular signal with small width.

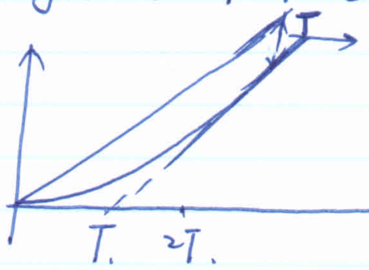
• Ramp Signal Response

$r(t) = t \qquad R(s) = \frac{1}{s^2}$

$Y(s) = \Phi(s)\, R(s)$

$= \frac{1}{Ts+1} \cdot \frac{1}{s^2} = \frac{1}{s^2} - \frac{T}{s} + \frac{T^2}{Ts+1}$

$y(t) = t - T(1 - e^{-\frac{t}{T}}) \quad t \geq 0$



steady state error.

The response = steady + dynamic.

$\downarrow$ steady: t-T.

$\downarrow$ dynamic: goes to 0.

Steady state error exists. When 1st-order system tries to track the ramp signal, ~~so~~ the variation trend is same. steady state error exists.

✗ : Reducing T : Quick.

Accurate (error

* : • With 0 initial condition. unit impulse response is the T.F.

- We always use IR to get the closed-loop transfer function
- In practical, impossible to get ideal impulse signal. Instead of that, using a rectangular signal with small width.

$\frac{2s+1}{(s+1)(s^2+1)(s+2)}$

* $\boxed{\text{Gain of the system}}$ / DC gain

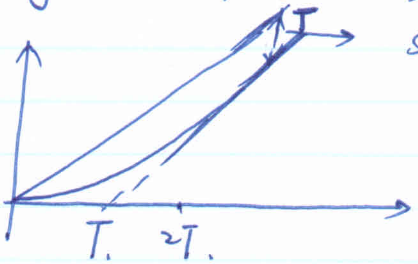
- Ramp Signal Response

$$r(t) = t \qquad R(s) = \frac{1}{s^2}$$

$$Y(s) = \Phi(s) R(s)$$

$$= \frac{1}{Ts+1} \cdot \frac{1}{s^2} = \frac{1}{s^2} - \frac{T}{s} + \frac{T^2}{Ts+1}$$

$$y(t) = t - T(1 - e^{-\frac{t}{T}}) \quad t \geq 0$$



The response = steady + dynamic.

$$\underset{t-T.}{\downarrow} \qquad \underset{\text{goes to } 0.}{\downarrow}$$

steady state error exists. When 1st-order system tries to track the ramp signal, ~~so~~ the variation trend is same. steady state error exists.

* : Reducing T : Quick.

Accurate (error

- Acceleration signal response

$$r(t) = \frac{1}{2}t^2 \qquad R(s) = \frac{1}{s^3}$$

$$Y(s) = \Phi(s)R(s) = \frac{1}{Ts+1}\frac{1}{s^3}$$

$$= \frac{1}{s^3} - \frac{1}{s^2} + \frac{T^2}{s} - \frac{T^3}{Ts+1}$$

$$y(t) = \frac{1}{2}t^2 - Tt + T^2(1 - e^{-t/T}) \quad t \geq 0$$

$$e(t) = r(t) - y(t) = Tt - T^2(1 - e^{-t/T}) \quad t \geq 0$$

$$e(t) \longrightarrow \infty \text{ as } t \to \infty$$

* 1st-order system cannot track acceleration signal.

*

$r(t)$	$y(t)$	
$\delta(t)$	$\frac{1}{T}e^{-t/T}$	
$\uparrow D$	$\uparrow D$	
$1(t)$	$1 - e^{-t/T}$	no steady state error
$\uparrow D$	$\uparrow D$	
t	$t - T(1 - e^{-t/T})$	constant steady state error.
$\uparrow D$	$\uparrow D$	
$\frac{1}{2}t^2$	$\frac{1}{2}t^2 - Tt + T^2(1 - e^{-t/T})$	cannot track.

The response to a signal's ~~integral~~ derivative can be calculated by solving the derivative ~~integral~~ of the original signal.

textbook Section 4.4

§ 7.3. Time domain analysis of 2nd-order System

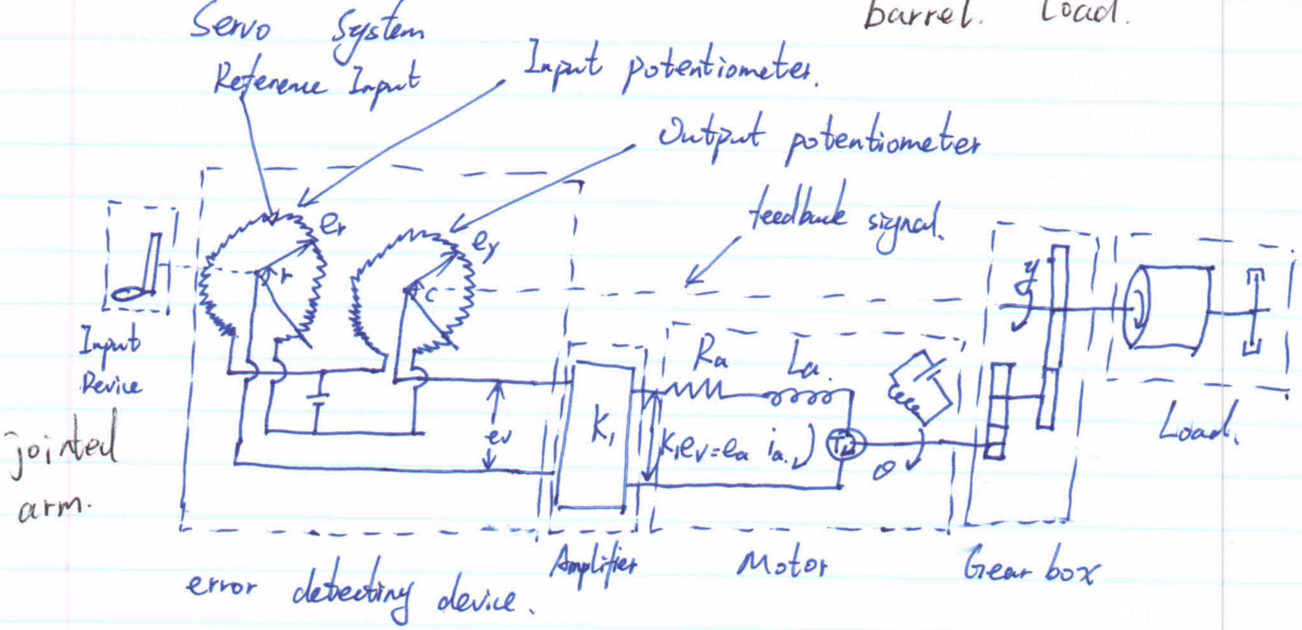
Review of §7.2.
- Time domain analysis of 1st-order systems
- Impulse, step, ramp, acceleration response
- characteristics.

- Start from an example.

power assisted steering

artillery servo system

barrel. Load.



- Controller signal: Load angle $y(t)$, Load is assembled with a potentiometer in a same axel.

- Input signal: Angle of Input potentiometer

- Error signal: error between $r(t)$ & $y(t)$

⟹ Output Voltage : $e_r = K_0 r$ $e_y = K_0 y$

⟹ Amplifier: Input: $e_v = e_r - e_y = K_0(r-y)$
Output: $e_a = K_1 e_v$

⟹ DC motor armature Voltage-Current Equation

$$L_a \frac{di_a}{dt} + R_a i_a + e_b = e_a$$

e_b : Back- EMF $e_b = K_3 \frac{d\theta}{dt}$ proportional to velocity.
K_3: Back- EMF parameter

L_a: Very small. can be neglected.

$$R_a i_a = e_a - e_b$$
$$= e_a - K_3 \frac{d\theta}{dt}$$

⟹ DC motor Torque equation

$$J_0 \frac{d^2\theta}{dt^2} + b_0 \frac{d\theta}{dt} = M = K_2 i_a.$$

$$\dot{\theta}(s) = \frac{M(s)}{J_0 s + b_0}$$

$$J_0 s \dot{\theta}(s) + b_0 \dot{\theta}(s) = K_2 i_a(s).$$

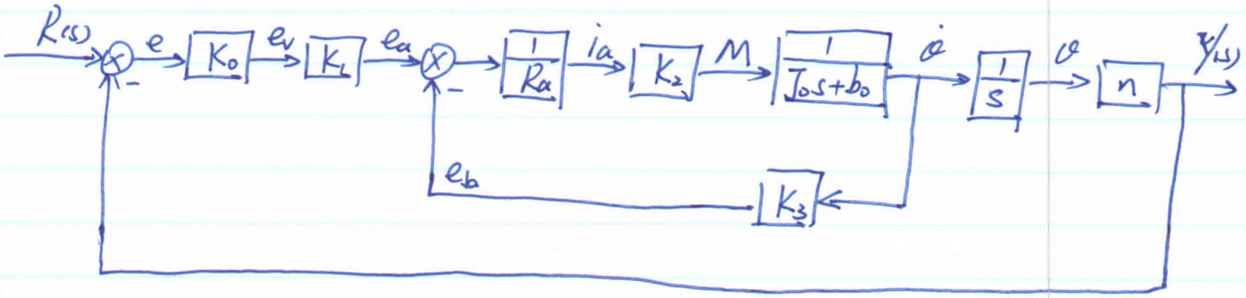
~~$\theta(s) = \frac{M(s)}{J_0 s^2 + b_0 s}$~~

J_0 : Calculated Inertia
b_0 : Calculated friction parameter
K_2: DC motor torque ~~param~~ constant

⟹ Gear Reduction ratio : n

$$y = n \cdot \theta$$

⇒ Block Diagram



Open-loop T.F. $G(s) = \dfrac{K_m}{s(T_m s + 1)}$

$$K_m = \frac{n K_0 K_1 K_2}{R_a b_0 + K_2 K_3}$$

$$T_m = \frac{R_0 J_0}{R_a b_0 + K_2 K_3}$$

Closed-loop T.F. $\Phi(s) = \dfrac{Y(s)}{R(s)} = \dfrac{K_m}{T_m s^2 + s + K_m}$

To be general: $\Phi(s) = \dfrac{Y(s)}{R(s)} = \dfrac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$ closed-loop

open-loop: $G(s) = \dfrac{\omega_n^2}{s(s + 2\zeta\omega_n)}$

ω_n: No damping oscillation frequency/natural $\omega_n = \sqrt{\dfrac{K_m}{T_m}}$

ζ: damping ratio

$$\zeta = \frac{1}{2\sqrt{T_m K_m}}$$

⇒ Characteristic equation: $s^2 + 2\zeta\omega_n s + \omega_n^2 = 0$

⇒ Poles: $s_{1,2} = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1}$ $\quad \zeta \geq 1$

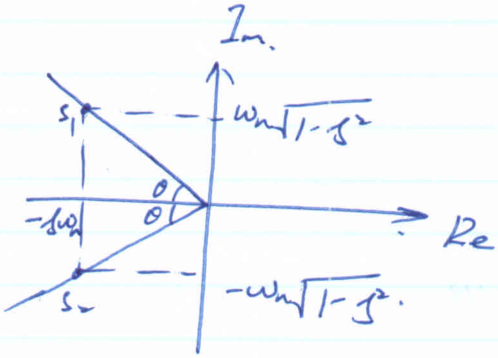
$s_{1,2} = -\zeta\omega_n \pm j\,\omega_n\sqrt{1-\zeta^2}$ $\quad 0 < \zeta < 1$

§ 7.3.1 2nd-order system step response. (4.5.4.6)

1. Under damped: $0 < \zeta < 1$

poles: Conjugate

$$s_{1,2} = -\zeta\omega_n \pm j\omega_n\sqrt{1-\zeta^2}$$



$$\Phi(s) = \frac{Y(s)}{R(s)} = \frac{\omega_n^2}{(s+\zeta\omega_n+j\omega_d)(s+\zeta\omega_n-j\omega_d)}$$

ω_d: damped oscillation frequency $\omega_d = \omega_n\sqrt{1-\zeta^2}$

$$Y(s) = \frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}\frac{1}{s} = \frac{1}{s} - \frac{s+\zeta\omega_n}{(s+\zeta\omega_n)^2+\omega_d^2} - \frac{\zeta\omega_n}{(s+\zeta\omega_n)^2+\omega_d^2}$$

$$\mathcal{L}^{-1}\left\{\frac{s+\zeta\omega_n}{(s+\zeta\omega_n)^2+\omega_d^2}\right\} = e^{-\zeta\omega_n t}\cos\omega_d t.$$

$$\mathcal{L}^{-1}\left\{\frac{\omega_d}{(s+\zeta\omega_n)^2+\omega_d^2}\right\} = e^{-\zeta\omega_n t}\sin\omega_d t$$

0 initial condition.

cos, sin oscillate
the envelop of these
two terms is decaying.

~~$y(t) = 1 - e^{-\zeta\omega_n t}\frac{1}{\sqrt{1-\zeta^2}}$~~

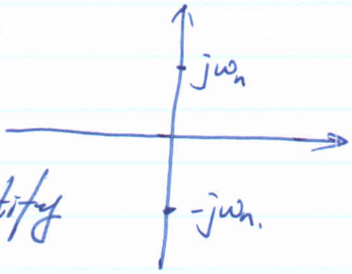
$$y(t) = 1 - e^{-\zeta\omega_n t}\left(\cos\omega_d t + \frac{\zeta\omega_n}{\omega_d}\sin\omega_d t\right)$$

$$= 1 - e^{-\zeta\omega_n t}\left(\cos\omega_d t + \frac{\zeta}{\sqrt{1-\zeta^2}}\sin\omega_d t\right)$$

$\Leftarrow \sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta \Leftarrow$

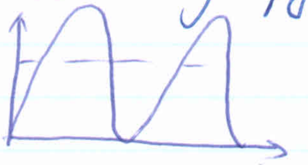
$$= 1 - \frac{e^{-\zeta\omega_n t}}{\quad}\sin(\omega_d t + \theta) \quad \theta = \quad \sqrt{1-\zeta^2}$$

2. $\zeta = 0 \Rightarrow \quad s_{1,2} = \pm j\omega_n.$



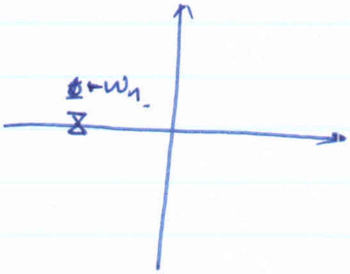
Through the place of poles, we can justify if there is damping.

$y(t) = 1 - \cos\omega_n t \quad t \geq 0$



3. Critically - damped $\quad \zeta = 1$

$s_{1,2} = -\omega_n.$

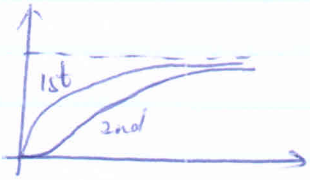


step:

$$Y(s) = \frac{\omega_n^2}{(s+\omega_n)^2} \cdot \frac{1}{s}$$

$$= \frac{1}{s} - \frac{\omega_n}{(s+\omega_n)^2} - \frac{1}{(s+\omega_n)}$$

$y(t) = 1 - e^{-\omega_n t}(1 + \omega_n t) \quad t \geq 0.$ Monotonically increasing.



Slope at $t = 0$: $\frac{dy}{dt} = \omega_n^2 t e^{-\omega_n t}$

$= 0$

1st - order system slope $= \frac{1}{T}$

> Difference.

4. Over - damped $\quad \zeta > 1$

$s_{1,2} = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1}$ two different negative real roots

$$Y(s) = \frac{\omega_n^2}{(s+\zeta\omega_n+\omega_n\sqrt{\zeta^2-1})(s+\zeta\omega_n-\omega_n\sqrt{\zeta^2-1})} \cdot \frac{1}{s}$$

~~$y(t) = 1 - \frac{1}{2\sqrt{\zeta^2-1}(\zeta-\sqrt{\zeta^2-1})} e^{-(\zeta-\sqrt{\zeta^2-1})\omega_n t} + \frac{1}{2\sqrt{\zeta^2-1}(\zeta+\sqrt{\zeta^2-1})} e^{-(\zeta+\sqrt{\zeta^2-1})\omega_n t}$~~

$=$ ~~$1 - e^{-\zeta\omega_n t}(\cos\omega_d t$~~

$$\checkmark = \boxed{1 - e^{-\zeta\omega_n t} \frac{1}{\sqrt{1-\zeta^2}} \sin(\omega_d t + \theta) \quad t \geq 0}$$

~~$\theta = \arctan\frac{\sqrt{1-\zeta^2}}{\zeta}$~~ ~~$/ \ \theta = \arccos\zeta$~~

for underdamped system

⇒ Two parts : steady state
Transient state

⇒ Damped oscillation . frequency ω_d → damped frequency.

⇒ Decay rate → $\zeta\omega_n$ ↑ far away from image axis

steady state : • The value after transient state decays to 0.

• Same as reference signal.

✗ : 2nd-order system, underdamped. step , no steady state error.

2. No damping $\zeta = 0$

$$y(t) = 1 - \cos\omega_n t \quad t \geq 0$$

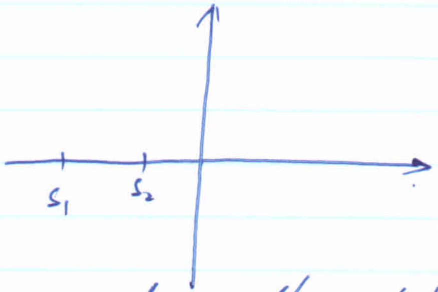
time response equal-amplitude oscillation. frequency $\underset{\substack{\downarrow \\ \text{no damping} \\ \text{frequency}}}{\omega_n}$.

$$y(t) = 1 - \frac{1}{2\sqrt{\zeta^2-1}\,(\zeta-\sqrt{\zeta^2-1})}\, e^{-(\zeta-\sqrt{\zeta^2-1})\omega_n t}$$

$$+ \frac{1}{2\sqrt{\zeta^2-1}\,(\zeta+\sqrt{\zeta^2-1})}\, e^{-(\zeta+\sqrt{\zeta^2-1})\omega_n t} \qquad t \geq 0$$

2 exponential decaying terms

$s_1 = -(\zeta+\sqrt{\zeta^2-1})\omega_n$ $\qquad$ $s_2 = -(\zeta-\sqrt{\zeta^2-1})\omega_n$.



Decaying rate of s_1 is larger than that of s_2

$\zeta >> 1$, s_1 can be neglected

So we can regard it as 1st-order system

$$y(t) \simeq 1 - \frac{1}{2\sqrt{\zeta^2-1}\,(\zeta-\sqrt{\zeta^2-1})}\, e^{-(\zeta-\sqrt{\zeta^2-1})\omega_n t}$$ approximated by 1st-order system

$$T.F. \simeq \frac{Y(s)}{R(s)} \simeq \frac{-s_2}{s-s_2} = \frac{1}{Ts+1} \qquad T = \frac{1}{-s_2}$$

We can do this if $\zeta > 2$

5. Negative damping $-1 < \zeta < 0$

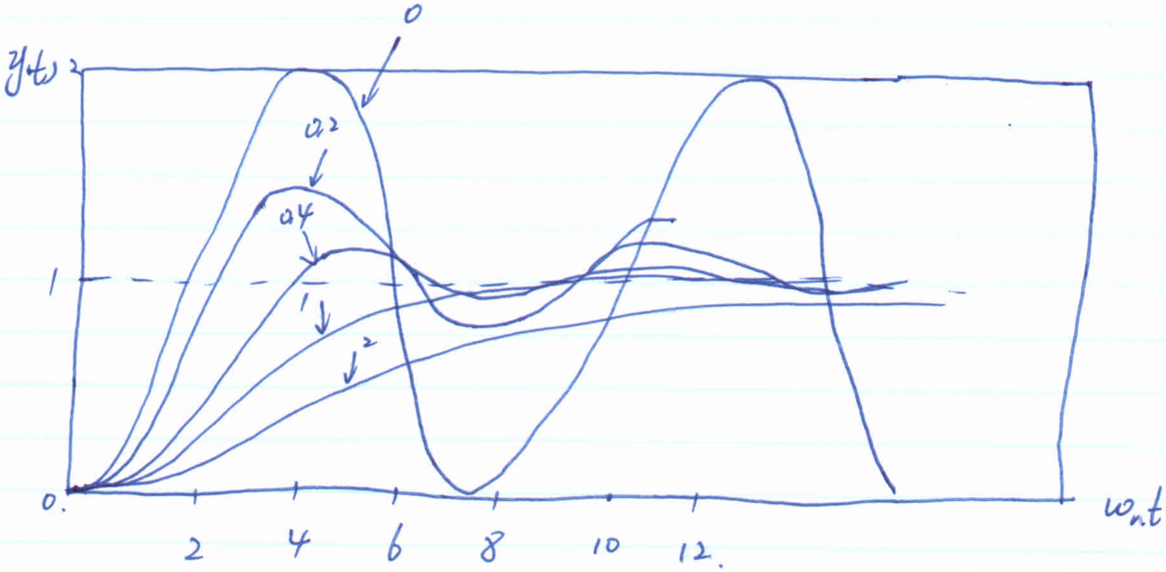
$$y(t) = 1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}} \sin(\omega_d t + \theta)$$

$\omega_d = \omega_n\sqrt{1-\zeta^2}$

$\theta = \arctan\frac{\sqrt{1-\zeta^2}}{\zeta} < 0$

Diverged oscillation

$0 < \zeta \leq 2$.



§ 7.3.2 Dynamic Performance (under damped)

Dynamic ~~procedure~~ process ⟶ Transient process

: Under the input signal, system changes from the initial state to the final steady state.

Good or Bad: Quick Steady

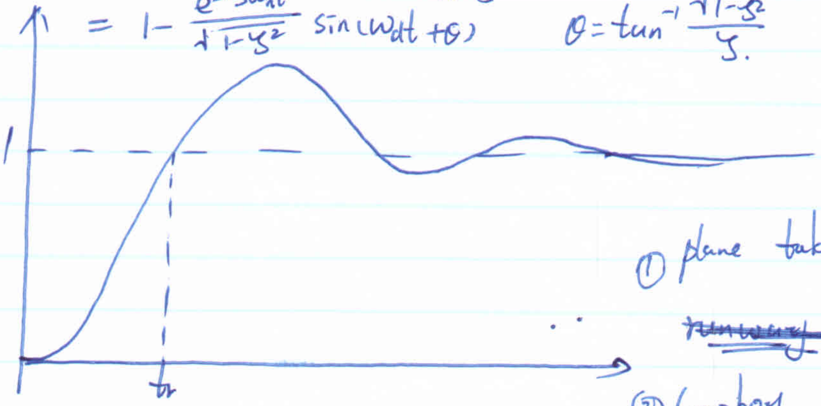
Always use step response to justify

$$1-e^{-\zeta\omega_n t}\left(\cos\omega_d t+\frac{\zeta}{\sqrt{1-\zeta^2}}\sin\omega_d t\right)$$

$$=1-\frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\sin(\omega_d t+\theta) \qquad \theta=\tan^{-1}\frac{\sqrt{1-\zeta^2}}{\zeta}$$

Typical response.

under damped.



① plane take off

~~runway start~~

② Cowboy. duel.

① Rising time t_r: The first time The step response reaches the steady state value $y(\infty)$ ~~for the first time~~

Quick

For over damped system, consider time from $0.1\,y(\infty)$ to $0.9\,y(\infty)$

② Peak time t_p: The first time $y(t)$ reaches the peak value.

③ Overshoot σ_p:

Steady

$$\sigma_p \triangleq \frac{y(t_p)-y(\infty)}{y(\infty)}\times 100\%$$

* No overshoot for 1st-order system & over damped 2nd-order

④ Settling time t_s: The time instant, $y(t)$ reaches & keeps within the range of $y(\infty) \pm 5\% / 2\%$.

$\boxed{\text{Quick.}}$

$$|y(t) - y(\infty)| \le \Delta y(\infty) \quad t \ge t_s$$

$$\Delta = 5\% \;/\; 2\%$$

※. After t_s, we consider the dynamic process is over.

⑤ Oscillation times N: In dynamic process ($t \le t_s$), the oscillation times of $y(t)$

How many times $y(t)$ go across $y(\infty)$. $1/2$.

$\boxed{\text{steady}}$

Most useful two terms are σ_p & t_s

We forcus on the under-damped system —

$\Longrightarrow$ Calculation of performance indices.

- Definitly, we can calculate those indices from the step response. But, I.L.F. complicated calculation. Don't use.
- We want to know, if we can calculate from the parameters in T.F.

① t_r

By definition $y(t)=1$. $t=t_r$

$$y(t_r)=1-\frac{e^{-\zeta\omega_n t_r}}{\sqrt{1-\zeta}}\sin(\omega_d t_r+\theta)=1$$

$$\underset{\overset{\neq}{0.}}{\underline{e^{-\zeta\omega_n t_r}}}\ \underset{\overset{\parallel}{0.}}{\underline{\sin(\omega_d t_r+\theta)=0}}$$

$$\sin(\omega_d t_r+\theta)=0 \qquad \Rightarrow \qquad \boxed{t_r=\frac{\pi-\theta}{\omega_d}=\frac{\pi-\theta}{\omega_n\sqrt{1-\zeta^2}}}$$

$$\theta=\arctan\frac{\sqrt{1-\zeta^2}}{\zeta}.$$

$\Rightarrow$. $\boxed{\zeta \text{ fixed. Large } \omega_n \longrightarrow \text{short } t_r.}$ $\boxed{\omega_n \text{ fixed } \zeta\downarrow\ \theta\uparrow \quad t_r\downarrow.}$

Quickness $\propto$ ω_n.

② t_p

Derivative. $\left.\frac{dy(t)}{dt}\right|_{t=t_p}=0.$

$$\zeta\omega_n e^{-\zeta\omega_n t_p}(\cancel{\cos\omega_d t_p}+\frac{\zeta}{\sqrt{1-\zeta^2}}\sin\omega_d t_p)+e^{-\zeta\omega_n t_p}(\omega_d\sin\omega_d t_p-\frac{\zeta\omega_d}{\sqrt{1-\zeta^2}}\cancel{\cos\omega_d t_p})=0$$

$$(\zeta\omega_n e^{-\zeta\omega_n t_p}\frac{\zeta}{\sqrt{1-\zeta^2}}+\omega_d\ e^{-\zeta\omega_n t_p})\sin\omega_d t_p=0$$

$$\omega_n e^{-\zeta\omega_n t_p}(\frac{\zeta^2}{\sqrt{1-\zeta^2}}+\sqrt{1-\zeta^2})\sin\omega_d t_p=0$$

$$\boxed{=\frac{\omega_n}{\sqrt{1-\zeta^2}}e^{-\zeta\omega_n t_p}\ \sin\omega_d t_p=0}$$

$$\Rightarrow \omega_d t_p = 0,\ \underset{\substack{\downarrow \\ \text{take this}}}{\underline{\pi}},\ 2\pi,\ 3\pi,$$

$$\boxed{t_p = \frac{\pi}{\omega_d} = \frac{\pi}{\omega_n\sqrt{1-\zeta^2}}}$$

③ σ_p.

$$\sigma_p = \frac{y(t_p) - y(\infty)}{y(\infty)} \times 100\% = -e^{-\zeta\omega_n t_p}\left(\cos\omega_d t_p + \frac{\zeta}{\sqrt{1-\zeta^2}}\sin\omega_d t_p\right)$$

$$t_p = \frac{\pi}{\omega_n\sqrt{1-\zeta^2}}$$

$$\boxed{\sigma_p = e^{-\frac{\zeta}{\sqrt{1-\zeta^2}}\pi} \times 100\%}$$

ω_n has nothing to do with σ_p.

σ_p is only determined by ζ.

$\zeta \longrightarrow \sigma_p$.

Desired $\sigma_p \longrightarrow \zeta$.

$\downarrow \zeta = 0{,}4 \sim 0{,}8$.

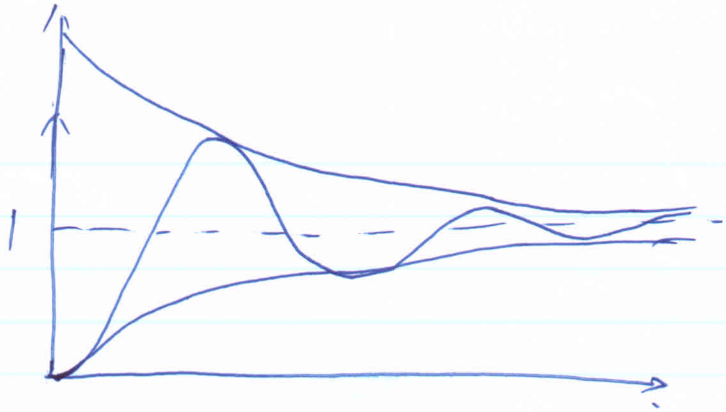
$\uparrow \sigma_p = 25\% \sim 2{,}5\%$.

④ t_s

$$y(t) = 1 - \frac{1}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t} \sin\left(\omega_d t + \arctan\frac{\sqrt{1-\zeta^2}}{\zeta}\right)$$

A decaying sinusoidal oscillation

Curve $1 \pm \frac{1}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t}$ are the envelopes



$$|y(t) - y(\infty)| \le \Delta y(\infty) \qquad t \ge t_s$$

$$\left| \frac{1}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t} \sin(\omega_d t + \arctan\frac{\sqrt{1-\zeta^2}}{\zeta}) \right| \le \Delta$$

Complicated. So we use the envelope.
If - envelope $\le \Delta$, than the response $\le \Delta$

$$\Rightarrow \left| \frac{1}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t} \right| \le \Delta \qquad t \ge t_s$$

$$t_s = \frac{1}{\zeta\omega_n} \ln \frac{1}{\Delta\sqrt{1-\zeta^2}}$$

~~$\Delta = 0.05$~~

$$\boxed{\begin{array}{ll} t_s = \dfrac{4 + \ln\frac{1}{\sqrt{1-\zeta^2}}}{\zeta\omega_n} & \Delta = 0.02 \\ t_s = \dfrac{3 + \ln\frac{1}{\sqrt{1-\zeta^2}}}{\zeta\omega_n} & \Delta = 0.05 \end{array}}$$

$0 < \zeta < 0.9$. $\ln\frac{1}{\sqrt{1-\zeta^2}}$ small.

$$\boxed{\begin{array}{ll} t_s \approx \dfrac{4}{\zeta\omega_n} & \Delta = 0.02 \\ t_s \approx \dfrac{3}{\zeta\omega_n} & \Delta = 0.05 \end{array}}$$

$\zeta\downarrow$. responses quick, but ~~enter~~ enters steady state slowly.
$\zeta\downarrow$ not steady

$\zeta\omega_n$: Distance between the closed-loop pole to image axis

For award $\rightarrow$ Quick

③ N.

The oscillation times between $0<t<t_s$

Frequency $\omega_d = \omega_n\sqrt{1-\zeta^2}$ $\quad T_d = \frac{2\pi}{\omega_d}$

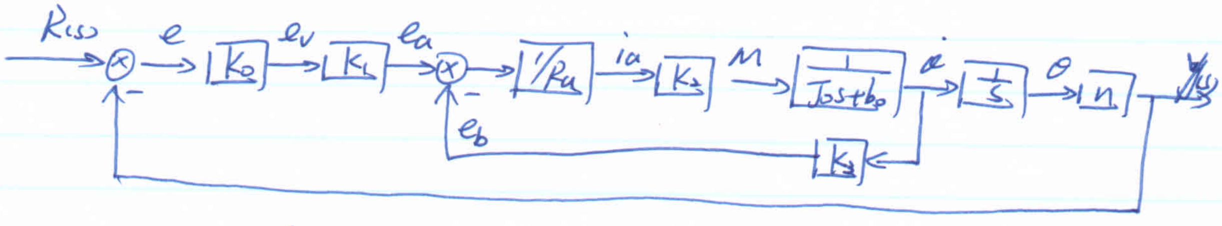
$$N = \frac{t_s}{T_d}$$

$$N = \begin{cases} \dfrac{2\sqrt{1-\zeta^2}}{\pi\zeta} & \Delta = 0{,}02 \\ \dfrac{1{,}5\sqrt{1-\zeta^2}}{\pi\zeta} & \Delta = 0{,}05 \end{cases}$$

take integer

- From above, satisfying performance. appropriate ζ and ω_n should be chosen.
- $\omega_n \uparrow$ $\underline{\underline{t_r \downarrow}}$ response speed.
- $\zeta \uparrow$ steady. reduce σ_p & N.
  $\underline{\underline{t_r \uparrow}}$

Example 1: Servo system



$$G(s) = \frac{K_m}{s(T_m s + 1)} \qquad \Phi(s) = \frac{K}{T_m s^2 + s + K}$$

$$K_m = \frac{n k_0 k_1 k_2 k_3}{R_a b_0 + k_2 k_3}$$

$$T_m = \frac{R_a J_0}{R_a b_0 + k_2 k_3}$$

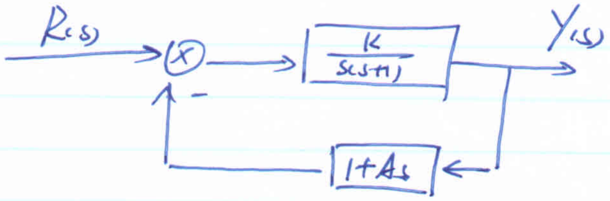
$\omega_n = \sqrt{K/T_m}$ $\qquad$ $\zeta = 1/2\sqrt{T_m K}.$

$\omega_n \uparrow$ raise $K_2 \rightarrow K\uparrow \rightarrow \zeta\downarrow$.

Quick. $\qquad\qquad$ not steady.

- So. when designing the system, pay attention to reducing T_m. like reduce J, a, + and choose large K_2, K_3
  motor.
- Otherwise, trade-off exists $\qquad$ reducing T_m + enlarging K

Example 2:



Require: $\sigma_p = 20\%$, $t_p = 1$. $\qquad$ K, A?
t_r, t_s, N?

$$\frac{Y(s)}{R(s)} = \frac{K}{s^2 + (1+KA)s + K}.$$

$\omega_n = \sqrt{K}.$ $\qquad$ $2\zeta\omega_n = 1 + KA.$

$$\sigma_p = e^{-\frac{\zeta}{\sqrt{1-\zeta^2}}\pi}$$

$$\frac{\pi\zeta}{\sqrt{1-\zeta^2}} = \ln\frac{1}{\sigma_p} = 1.61 \Rightarrow \zeta = 0.456.$$

$$\omega_n = \frac{\pi}{t_p\sqrt{1-\zeta^2}} = 3.53 \qquad t_p = \frac{\pi}{\omega_n\sqrt{1-\zeta^2}}$$

$K = \omega_n^2 = 12.5.$

$$A = \frac{2\zeta\omega_n - 1}{K} = 0.178.$$

$$t_r = \frac{\pi-\theta}{\omega_n\sqrt{1-\zeta^2}} = 0.65$$

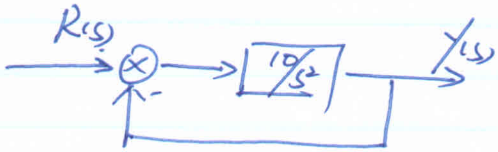
$$\theta = \arctan\frac{\sqrt{1-\zeta^2}}{\zeta} = 1.1$$

$$t_s = \frac{4+\ln\frac{1}{\sqrt{1-\zeta^2}}}{\zeta\omega_n} = 2.56 \qquad \Delta=0.02$$

$$= \frac{3+\ln\frac{1}{\sqrt{1-\zeta^2}}}{\zeta\omega_n} = 1.94 \qquad \Delta=0.05$$

$$N = \frac{2\sqrt{1-\zeta^2}}{\pi\zeta} = 1.42 = 1 \qquad \Delta=0.02.$$

Example 3:



step response Works normally?

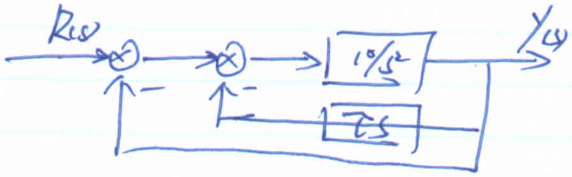
if require $\zeta=0.707$. how to improve.

$$\frac{Y(s)}{R(s)} = \frac{10}{s^2+10}. \qquad \zeta=0.$$

$y(t) = 1 - \cos(\sqrt{10}\,t)$. oscillation, no decaying. $\omega_n=\sqrt{10}$.

It can not reflect $r(t)=1$.

Plus. derivative negative feedback.



$$\frac{Y(s)}{R(s)} = \frac{10}{s^2 + 10\tau s + 10}$$

$$2\zeta\omega_n = 10\tau \quad \Rightarrow \quad \tau = \frac{2\zeta\omega_n}{10} = 0{,}447$$

$$\omega_n = \sqrt{10}$$

- Oscillation ⟶ Decaying oscillations
- Negative derivative feedback → raises damping.
- σ_p : 100% → 4.3%.

§8 Improve System Performance

Review §7.

- step response of 2nd-order system
- Performance indices & calculations
  t_r. t_p. σ_p. t_s, N.

- Using time domain response analysis can solve for the performance indice.
- If not satisfied. we need to improve.

too slow.

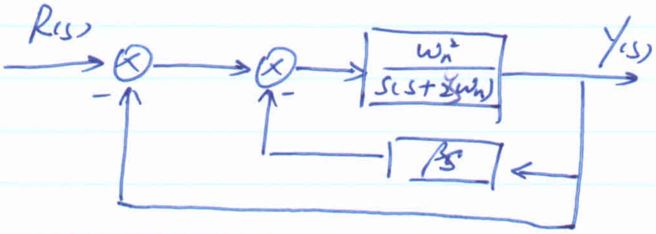
large overshoot.

§ 8.1 Speed feedback.

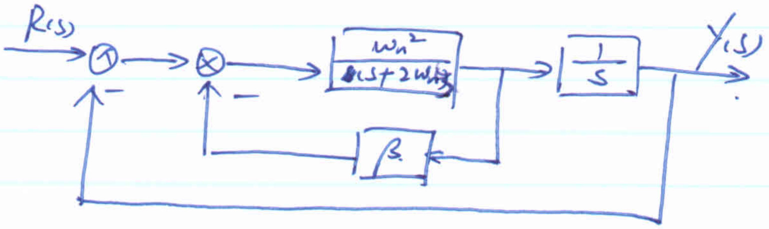
$$G(s) = \frac{\omega_n^2}{s(s+2\zeta\omega_n)}$$ typical. 2nd-order open-loop T.F.

If we can detect the speed of $y(t)$, and feed the speed back to the system input, then compare with the error signal, the whole closed-loop system is the one with speed feedback.

Speed: not only speed, but the derivative



⟺.



For this kind of servo system, output ⟶ angle.

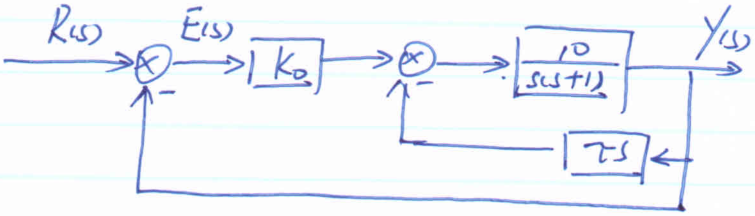
speed detecting genetor ⟶ voltage ∝ angular velocity.

Closed-loop T.F: $\Phi(s) = \frac{Y(s)}{R(s)} = \frac{\omega_n^2}{s^2+(2\zeta\omega_n+\omega_n^2\beta)s+\omega_n^2}$

$$= \frac{\omega_n^2}{s^2+2\bar{\zeta}\omega_n s+\omega_n^2}$$

$\Longrightarrow$ $\bar{\zeta} = \zeta + \frac{1}{2}\beta\omega_n$. Damping ratio raised.

Example:



Require: $\sigma_p = 16.3\%$ $t_p = 1$ s.

? K_0, τ.

Open loop: $G(s) = \dfrac{10K_0}{s^2 + (1+10\tau)s}$

Closed loop: $\Phi(s) = \dfrac{10K_0}{s^2 + (1+10\tau)s + 10K_0}$

$$\omega_n^2 = 10K_0$$
$$2\zeta\omega_n = 1 + 10\tau$$

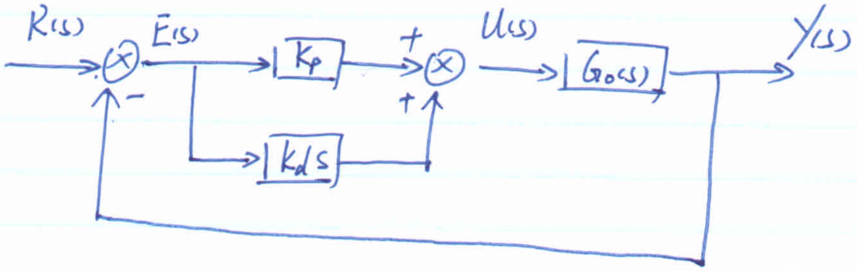
$$\begin{cases} \sigma_p = e^{-\zeta\pi/\sqrt{1-\zeta^2}} = 0.163 \\ t_p = \dfrac{\pi}{\omega_n\sqrt{1-\zeta^2}} = 1 \end{cases} \Rightarrow \begin{cases} \zeta = 0.5 \\ \omega_n = 3.63 \end{cases}$$

$\Rightarrow K_0 = 1.32 \qquad \tau = 0.263$

Speed feedback is kind of modifying the system

§ 8.2 PD control (proportional + derivative)

Classical PID. (be used in lab)



K_p: Proportional Gain.

K_d: Derivative Gain.

T.F. of PD controller: $G_c(s) = K_p + K_d s = K_p(1 + \frac{K_d}{K_p}s)$

$= K_p(1 + \tau s)$

Why derivative?

- Derivative can predict the variation of the error signal
- Correct the control signal before the variation happens

Take 2nd-order system as an example:

$$G_0(s) = \frac{K_0}{s(Ts+1)}$$

No PD: $\Phi(s) = \frac{K_0}{Ts^2+s+K_0}$ $\Rightarrow$ $\omega_n = \sqrt{\frac{K_0}{T}}$

$\zeta = \frac{1}{2\sqrt{K_0 T}}$

With PD: $G_c(s) = K_p(\frac{K_d}{K_p}s+1) = K_p(\tau s+1)$

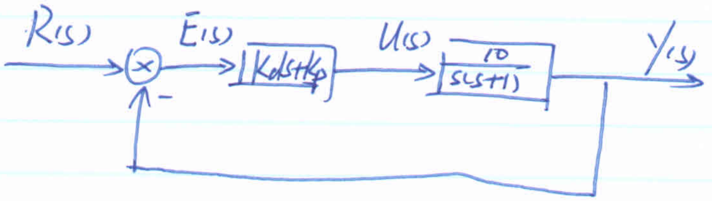
$$\bar{\Phi}(s) = \frac{K_0K_p(\tau s+1)}{Ts^2+(K_0K_p\tau+1)s+K_0K_p}$$

$$\bar{\omega}_n = \sqrt{\frac{K_0K_p}{T}} \qquad \bar{\zeta} = \frac{K_0K_p\tau+1}{2\sqrt{TK_0K_p}}$$

$\Rightarrow$ Approporiatly choosing K_p, K_d, can raise ω_n and ζ.

Quickness and steady can be improved

Example:



Require: $\sigma_p \le 16\%$ $t_s \le 4s$ $(\Delta=0.02)$

? K_p K_d

No PD: $\Phi(s) = \frac{10}{s^2 + s + 10}$

$\omega_n = \sqrt{10} = 3{,}16.$ $\zeta = \frac{1}{2\sqrt{10}} = 0{,}158$

$\sigma_p = 60\%$ $t_s = 8.$

With PD: $\Phi(s) = \frac{10K_p(\tau s + 1)}{s^2 + (10K_p\tau + 1)s + 10K_p}$ $\tau = \frac{K_d}{K_p}$

$\omega_n = \sqrt{10K_p}$ $\zeta = \frac{10K_p\tau + 1}{2\sqrt{10K_p}}$

S.T. $t_s \le 4$ $\sigma_p \le 16\%.$

$\Rightarrow \omega_n \ge 2$ $\zeta \ge 0{,}5$

$\Rightarrow K_p \ge 0{,}4$ $\tau \ge 0{,}25.$

$K_p = 0{,}4$ $\tau = 0{,}25$ $K_d = 0{,}1$

§9 Stability of Linear System

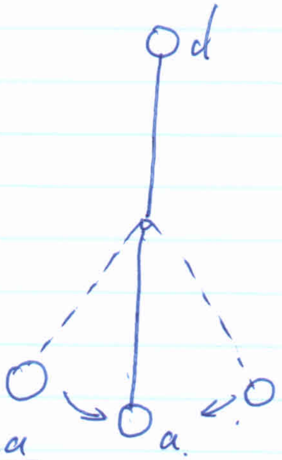
textbook Section 6.

Review §8 How to improve the performance?
Speed feedback.
PD control

- stability : The base for a system working normally.

Important

1. Basic Concepts.



Ex: Simple pendulum.

Only gravity & friction of air.

Two ~~equilibrium~~ balance point d.a

If let the initial position & each order derivative = 0.

Ball will stay at a or d.

If impulse input, after the impulse disappears. ① for a. the ball will go back. ② for d. cannot go back.

⇒ a. stable balance point
d. unstable balance point.

For a linear ordinary system,

closed-loop T.F. $\Phi(s) = \frac{Y(s)}{R(s)} = \frac{b_m s^m + b_{m-1} s^{m-1} + \cdots + b_1 s + b_0}{a_n s^n + a_{n-1} s^{n-1} + \cdots + a_1 s + a_0}$

The ODE to describe the movement is

$$a_n \frac{d^n y}{dt^n} + a_{n-1} \frac{d^{n-1} y}{dt^{n-1}} + \cdots + a_1 \frac{dy}{dt} + a_0 y$$

$$= b_m \frac{d^m r}{dt^m} + b_{m-1} \frac{d^{m-1} r}{dt^{m-1}} + \cdots + b_1 \frac{dr}{dt} + b_0 r$$

- 若 $r(t)=0$, if output $y(t)$ & each order derivative $=0$, then system will stay at $y(t)=0$

- $r(t) = \delta(t)$. ① impulse applied, if $y(t)$ can finally reach $y(t)=0$, and stay at $y(t)=0$, then we say <u>system is stable</u>

  ② If cannot reach $y(t)=0$, and cannot stay at $y(t)=0$, <u>say unstable</u>

The response according to impulse signal $\delta(t)$ is called

<u>Impulse response</u> $\longrightarrow k(t)$

The criterion to justify if a system is stable:

$$\lim_{t \to \infty} k(t) = 0 \quad \text{stable}$$

$$\neq 0 \quad \text{unstable}$$

More rigorous definition will be given later.

2. If and Only If conditions for the stability of linear system

- As mentioned, $k(t)$ can be used to evaluate the stability
- $\mathcal{L}[\delta(t)] = 1.$
- $k(t) = \mathcal{L}^{-1}[\Phi(s)] = \mathcal{L}^{-1}\left[\dfrac{k\prod_{i=1}^{m}(s-z_i)}{\prod_{j=1}^{q}(s-s_j)\prod_{k=1}^{r}(s^2+2\zeta_k\omega_{nk}s+\omega_{nk}^2)}\right]$

$$= \sum_{j=1}^{q} A_j e^{s_j t} + \sum_{k=1}^{r} B_k e^{-\zeta_k\omega_{nk}t}\cos(\omega_{nk}\sqrt{1-\zeta_k^2 \cancel{t}}\, t) + C_k e_{\cancel{k}}^{-\zeta_k\omega_{nk}t}\sin(\omega_{nk}\sqrt{1-\zeta_k^2}\, t)$$

$s_j\ (j=1, 2, \ldots, q)$ and $s_k = \cancel{\zeta} -\zeta_k\omega_{nk} \pm j\omega_{nk}\sqrt{1-\zeta_k^2}\ (k=1, 2, \ldots, r)$

$q+2r=n.$ are the closed-loop poles of the system

* ⇒ If and only if, all the closed-loop poles have negative real part, the impulse response will decay to 0. STABLE

For real ~~root~~ poles, be negative.

⇒ If one or more closed-loop poles have positive real parts the impulse response will diverge, UNSTABLE

⇒ If one or more closed-loop poles have zero real parts, the impulse response will converge to a constant or equal amplitude oscillation. CRITICALLY STABLE

only one zero pole is ok. two$^+$ → unstable.

⇒ Stability depends on the poles of closed-loop T.F.

$D(s) = a_n s^n + a_{n-1} s^{n-1} + \cdots + a_1 s + a_0 = 0$ Characteristic equation

Theorem: The If & Only If conditions for linear systems being stable is: The poles of closed-loop ~~to~~ characteristic equation ~~a~~ all have negative real parts. OR., all closed-loop poles are located at the left hand side.

- BIBO stable: Bounded Input Bounded Output stable
  If the input is bounded, output is bounded.

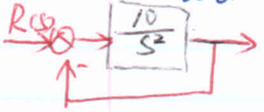
- Use step signal as an example ~~for~~ of bounded input, then the response has two parts:

  A bounded steady part → bounded

  A traisent part. → Decay to 0. negative real.

$$\frac{1}{TS+1}$$

- If 0 real part poles exist, i.e., poles on the image axis, the response will contain a non decaying oscillation, its bounded, so its BIBO stable, but not stable. (Cannot work normally).



$$\text{STABLE} \; \overset{\longrightarrow}{\not\longleftarrow} \; \text{BIBO STABLE}$$

⟹ If and only if all poles in the left hand side ~~of~~ of S plane, the system is stable; Otherwise, unstable

⟹ The stability of a closed-loop linear system depends on the poles' location on S plane which is determined by the system structure & parameters, does not rely on the input signal.

2. Routh Criteria

- It is easy to solve for the roots of 1st / 2nd even 3rd order characteristic equation. But for the high-order system, unless it can be rewritten as $(s-a_1)(s-a_2)\cdots$ otherwise, it's hard to ~~say~~ tell the stability.

Found by English mathematician Edward John Routh in 1876. To determine whether all the roots of the characteristic polynomial of linear system have negative real parts.

In 1895, German mathematician Adolf Hurwitz independently proposed a matrix method for the same goal.

So it's also called Routh-Hurwitz criterion.

- characteristic equation:

$$D(s) = a_n s^n + a_{n-1} s^{n-1} + \cdots + a_1 s + a_0 = 0$$

All the three following criteria satisfied:

① C_1 Coefficient for each term is not zero, no term missed.

② C_2 All coefficients are positive real numbers. (all negative ✓)

③ C_3 Use the coefficient to make Routh table, the first column. all greater than zero
parameters

Routh Table.

s^n	a_n	a_{n-2}	a_{n-4}	a_{n-6}
s^{n-1}	a_{n-1}	a_{n-3}	a_{n-5}	a_{n-7}
s^{n-2}	b_1	b_2	b_3	$\vdots$
$\vdots$	c_1	c_2	c_3	
s^2	$\vdots$	$\vdots$	$\vdots$	$\vdots$
s^1	$\vdots$	$\vdots$	$\vdots$	$\vdots$
s^0	—	—	—	—

$$b_1 = \frac{a_{n-1}a_{n-2} - a_n a_{n-3}}{a_{n-1}}$$

$$b_2 = \frac{a_{n-1}a_{n-4} - a_n a_{n-5}}{a_{n-1}}$$

$$b_3 = \frac{a_{n-1}\, a_{n-6} - a_n a_{n-7}}{a_{n-1}}$$

$$\vdots$$

$$c_1 = \frac{b_1 a_{n-3} - b_2 a_{n-1}}{b_1}$$

$$c_2 = \frac{b_1 a_{n-5} - a_{n-1} b_3}{b_1}$$ so on and so forth.

⟹: s^n, s^{n-1} ... s^0 are the indicator of the row, we should calculate till s^0.

⟹ Use a_n a_{n-1} b_1 c_1 ... to evaluate the stability

Characteristics:

① Multiply a positive real number to each term in one row does not affect the stability.

② For the first column, from the top to bottom, the number of sign changing equal to the number of poles which have positive real parts.

Example 1:

$D(s) = s^4 + 2s^3 + 3s^2 + 4s + 5 = 0.$

C_1 ✓

C_2 ✓

s^4	1	3	5
s^3	2	4	$\underline{0}$ → no term 0.
s^2	1	5	0
s^1	-6		
s^0	5		

not stable. 2 poles have positive real parts.

Example 2:

Special case 1: First element of one row is 0. Others are not 0 ort not all are zero.

$D(s) = s^4 + s^3 + 2s^2 + 2s + 5 = 0$

s^4	1	2	5
s^3	1	2	0
s^2	$\underline{0}$	5	

Cannot go

Let $s = \frac{1}{x}$ to construct a new equation of x. Obviously, the ~~roots wil~~ number of roots with positive real parts equal to the original one.

$$D(x) = 5x^4 + 2x^3 + 2x^2 + x + 1$$

x^4	5	2	1
x^3	2	1	0
x^2	-1	2	
x^1	5		
x^0	2.		

two not stable

- If $D(x)$ & $D(s)$ are the same, does not work.

We can multiply $(s+a)$ $a>0$ to the $D(s)$.

$\bar{D}(s) = D(s)(s+a)$ does not affect stability.

$\bar{D}(s) = D(s)(s+1)$

$$= s^5 + 2s^4 + 3s^3 + 4s^2 + 7s + 5$$

s^5	1	3	7
s^4	2	4	5
s^3	2	9	
s^2	-10	10	
s^1	11		
s^0	10		

two, not stable.

Special case 2: All elements in one row are 0

Use the row above the all-0 row to construct an auxiliary function $F(s)$, derivative, use the coefficients in $F'(s)$ to take place of all-0 row. and go on.

- ~~Auxiliary~~ Roots of the auxiliary function are part of the roots of the original function.
- Even-order function. a pair of roots.
- Solve for auxiliary function can obtain the unstable roots of the system.

$$D(s) = s^3 + 2s^2 + s + 2 = 0.$$

$$\begin{array}{lll} s^3 & 1 & 1 \\ s^2 & 2 & 2 \\ s^1 & 0 & 0. \end{array}$$

cannot continue.

$$F(s) = 2s^2 + 2 = 0.$$

$$F'(s) = 4s = 0.$$

$$\begin{array}{l|l|l} s^3 & 1 & 1 \\ s^2 & 2 & 2 \\ s^1 & 4 & 0 \\ s^0 & 2 & \end{array}$$

Even though. all positive, but

※ roots of $F(s) = 0$ $\underline{\underline{\pm j}}$.

not stable.

Example: $D(s) = s^6 + 4s^5 - 4s^4 + 4s^3 - 7s^2 - 8s + 10 = 0$

+ − sign. not stable, how many +?

s^6	1	-4	-7	10
s^5	4	4	-8	0
s^4	-5	-5	10	
s^3	(-1)	(-1)	2	
s^2	0 (-4)	0 (-2)		
s^1	-1/2 (-1)	2 (4)		
	-18			
s^0	4			

divided by 5.

auxiliary. $-s^4 - s^2 + 2 = 0$.

×2. $-4s^3 - 2s = 0$.

Changing ~~two times~~ twice.

$s_{1,2} = \pm j\sqrt{2}$.

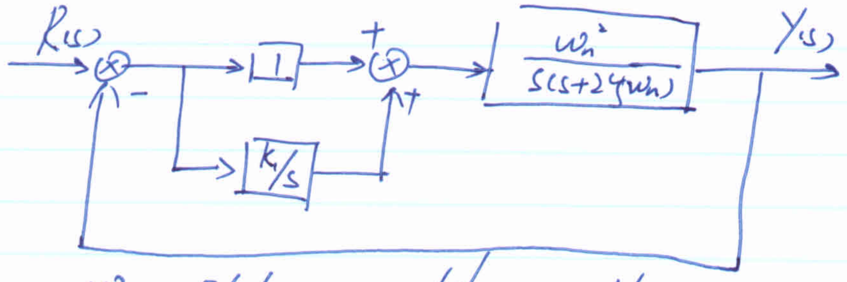
$s_{3,4} = \pm 1$.

two on the imag axis

3. Stability Conditions

① Solve for the range of a certain parameter

Example:



If $\zeta = 0.2$ $\omega_n = 86.6$. determine the range of K_1 to keep the system stable

Solution: $\Phi(s) = \frac{Y(s)}{R(s)} = \frac{\omega_n^2(s+K_1)}{s^3+2\zeta\omega_n s^2+\omega_n^2 s+K_1\omega_n^2}$

$$D(s) = s^3 + 34.6s^2 + 7500s + 7500K_1 = 0$$

Routh Table:

s^3	1	7500
s^2	34.6	$7500K_1$
s^1	$\frac{34.6\times7500-7500K_1}{34.6}$	0
s^0	$7500K_1$	

$$\Rightarrow \begin{cases} 34.6\times7500-7500K_1 > 0 \\ \cancel{7500}K_1 > 0 \end{cases}$$

$$\Rightarrow \quad 0 < K_1 < 34.6.$$

Example: $D(s)=s^3+(\lambda+1)s^2+(\lambda+\mu-1)s+(\mu-1)=0$

Keep the system stable, λ & μ?

Solution: Routh Table:

$$\begin{array}{lll} s^3 & 1 & \lambda+\mu-1 \\ s^2 & \lambda+1 & \mu-1 \\ s^1 & \frac{\lambda(\lambda+\mu)}{\lambda+1} & 0 \\ s^0 & \mu-1 & \end{array}$$

$$\Rightarrow \begin{cases} \lambda+1 > 0. \\ \lambda(\lambda+\mu) > 0 \\ \mu-1 > 0 \end{cases} \Rightarrow \lambda>0 \ \& \ \mu>1$$

② Solve for the relationship between some parameters.

Example: Open loop transfer function, unit feedback.

$$G(s)=\frac{K}{s(Ts+1)(2s+1)} \quad (K>0,\ T>0)$$

(1) What's the relationship between K & T if the system is stable

(2) Critically stable, $\omega = 1$ rad/s. K & T?

Solution:

$$D(s)= s(Ts+1)(2s+1)+K$$
$$= 2Ts^3+(2+T)s^2+s+K=0$$

Routh Table:

$$\begin{array}{lll} s^3 & 2T & 1 \\ s^2 & 2+T & K. \\ s^1 & \frac{2+T-2KT}{2+T} & 0 \\ s^0 & K. & \end{array}$$

$$\Rightarrow \begin{cases} K>0 \\ T>0. \\ 2+T-2KT>0 \end{cases}$$

$K < \frac{1}{T} + \frac{1}{2}$ & $K>0$, $T>0.$

(2) $\omega = 1$ rad/s. D(s) has $\pm j$ as the root.

s^1 row $=0$ $\quad K = \frac{1}{T}+\frac{1}{2}.$

s^2 auxiliary function $\quad (2+T)s^2+K=0$

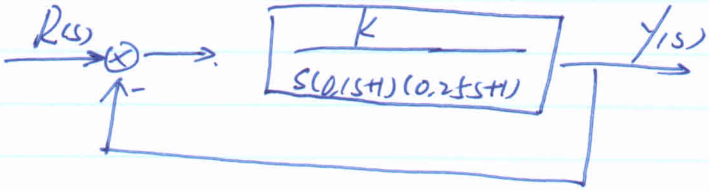
$\Rightarrow \quad s=\pm j\sqrt{\frac{K}{2+T}}$

$\Rightarrow \frac{K}{2+T}=1 \quad \Rightarrow \quad T=\frac{1}{2} \quad K=\frac{5}{2}$

OR substitute $s=\pm j$ into $D(s)=0$. let real & image parts all both 0.

③ Keep the system stable and let the closed-loop poles away from imag axis.

Example:



Requirement: all of the roots on the left hand side of $s=-1\pm j\omega$, range for K.

Solution: $D(s)= s^3+14s^2+40s+40K=0.$

Real(s) <-1 $\quad s=z-1$

$\Rightarrow$ Real(z) $<0.$

$z^3+11z^2+15z+(40K-27)=0$

Routh Table

$$\begin{array}{lll} z^3 & 1 & 15 \\ z^2 & 11 & 40K-27 \\ z^1 & \frac{11\times 5-(40K-27)}{11} & 0 \\ z^0 & 40K-27 & \end{array}$$

$$\Rightarrow \quad \begin{cases} 11\times 15 - (40K-27) > 0. \\ 40K-27 > 0 \end{cases}$$

$$\Rightarrow \quad 0.625 < K < 48.$$

4. Steady state Error

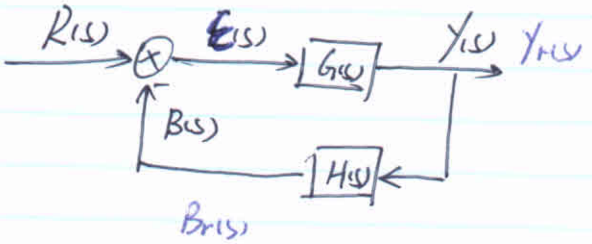
Review : Conditions for stabilizing system

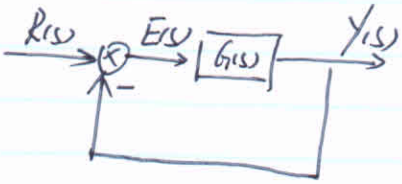
- Important: After the dynamic process, if the to-be-controlled variable can achieve the expected value
- Do not discuss the error caused by hardware.
- Only talk about steady state error when system is stable.

1) Error & Steady state error

Input $r(t)$ expected output $y_r(t)$
output $y(t)$

error $e(t) \triangleq y_r(t) - y(t)$ deviation: $\varepsilon(t) \triangleq r(t) - b(t)$.





We always like $B_r(s) = R(s)$

$$Y_r(s) = \frac{1}{H(s)} B_r(s) = \frac{1}{H(s)} R(s)$$

$$\underline{E(s)} = Y_r(s) - Y(s) = \frac{1}{H(s)} R(s) - \frac{1}{H(s)} B(s) = \frac{1}{H(s)} (R(s) - B(s))$$

error

$$= \frac{1}{H(s)} \underbrace{\varepsilon(s)}_{\text{deviation}} = \frac{1}{H(s)} \Phi_\varepsilon(s) \cdot R(s)$$

For unit feedback system $H(s)=1$. $\varepsilon(s)= E(s)$.

$e(t)= \varepsilon(t)$

$$\varepsilon(s) = \Phi_{\varepsilon}(s) \cdot R(s) = \frac{1}{1+G(s)} R(s)$$

Steady state error: $e_{ss} \triangleq \lim_{t\to\infty} e(t)$

Unit : $e_{ss} \triangleq \lim_{t\to\infty} \varepsilon(t)$

- Only $H(s)$ difference between $E(s)$ & $\varepsilon(s)$.
- We only discuss unit feedback system

2) Final Value Theorem & e_{ss} calculation

Typical open-loop T.F.

$$G(s) = \frac{K\prod_{i=1}^{m}(\tau_i s+1)}{s^{v}\prod_{j=1}^{k}(T_j s+1)\prod_{k=1}^{r}(T_k^2 s^2+2\zeta_k T_k s+1)}$$

K: Open loop Gain.

v: $\underbrace{\text{Open loop}}_{\text{number of}}$ integrals

Input - Error closed-loop transfer function

$$\Phi_{\varepsilon}(s) = \frac{\varepsilon(s)}{R(s)} = \frac{1}{1+G(s)} = \frac{s^{v}\prod_{j=1}^{q}(T_j s+1)\prod_{k=1}^{r}(T_k^2 s^2+2\zeta_k T_k s+1)}{s^{v}\prod_{j=1}^{q}(T_j s+1)\prod_{k=1}^{r}(T_k^2 s^2+2\zeta_k T_k s+1)+K\prod_{i=1}^{m}(\tau_i s+1)}$$

$$\varepsilon(s) = \Phi_{\varepsilon}(s)\, R(s)$$

If FVT conditions satisfied.

$$e_{ss} = \lim_{t\to\infty} e(t) = \lim_{s\to 0} s.\ \Phi_{\varepsilon}(s)\ R(s)$$

$$= \lim_{s\to 0.} s\cdot \frac{s^{v}\prod_{j=1}^{q}(T_j s+1)\prod_{k=1}^{r}(T_k^2 s^2+2\zeta_k T_k s+1)}{s^{v}\prod_{j=1}^{q}(T_j s+1)\prod_{k=1}^{r}(T_k^2 s^2+2\zeta_k T_k s+1)+K\prod_{i=1}^{m}(\tau_i s+1)}\cdot R(s).$$

$\Rightarrow$ S^{v} plays an important role.

unit step: $\frac{1}{s}$

ramp: $\frac{1}{s^2}$

acc: $\frac{1}{s^3}$

as long as the order of input signal $\leq v$.

$\Rightarrow e_{ss} = 0$.

So we call v in the closed-loop $\Phi_{\varepsilon}(s)$ "Degree of non-error" v-order non-error.

· v is also the number of integrals in Open-loop T.F.

※· e_{ss} is related to <u>$R(s)$</u> input.

① e_{ss} of unit step signal.

$$e_{ss} = \lim_{t\to\infty} e(t) = \lim_{s\to 0} sE(s) = \lim_{s\to 0} s\frac{1}{1+G(s)}R(s)$$

$$= \lim_{s\to\infty} \frac{1}{1+G(s)} = \frac{1}{1+\lim_{s\to\infty} G(s)}$$

e_{ss} depends on $\lim_{s\to\infty} G(s)$.

Define. K_p as the steady position error coefficient / position constant

$$K_p = \lim_{s\to 0} G(s)$$

$$K_p = \begin{cases} K & v=0, \\ \infty & v\geq 1 \end{cases}$$

$$\Rightarrow \quad e_{ss} = \begin{cases} \frac{1}{1+K} & v=0 \\ 0 & v\geq 1 \end{cases}$$

$\Rightarrow$ The system with $v=0$ under unit step, steady state error is a constant;
System with 1 & 2 under unit step. $e_{ss}=0$.

$\Rightarrow$ Open loop system: $v=0$ Type 0 system
$v=1$ Type I system
$v=2$ Type II system
$v>2$ Difficult to stabilize, rarely use.

② e_{ss} of ramp signal

For a stable ~~system~~ unit feedback system.
with $r(t) = t$
$R(s) = 1/s^2$

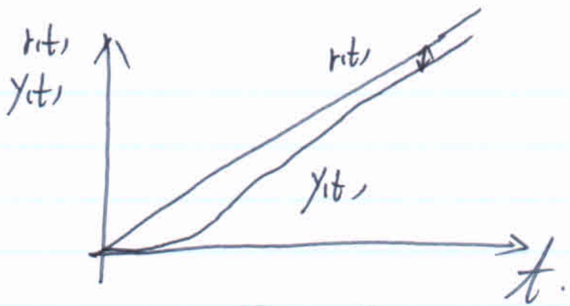
$$e_{ss} = \lim_{s\to 0} s \frac{1}{1+G(s)} \cdot \frac{1}{s^2} = \lim_{s\to 0} \frac{1}{sG(s)}$$

Define K_v as the steady velocity - error / coefficient
/velocity constant

$$K_v = \lim_{s\to 0} s\,G(s) = \begin{cases} 0 & v=0 \\ K & v=1 \\ \infty & v=2 \end{cases}$$

$$e_{ss} = \begin{cases} \infty & v=0 \\ \frac{1}{K_v} & v=1 \\ 0 & v=2 \end{cases}$$

⇒
- Type 0 system cannot track velocity signal, error will go ∞.
- Type I system can track, with steady state error a constant. Inverse proportional to open-loop gain K.
- Type II system can track velocity signal with no steady state error



Type I. ramp

③ e_{ss} with acceleration signal

For a stable unit feedback system, with.

$r(t) = \frac{1}{2}t^2$

$R(s) = \frac{1}{s^3}$

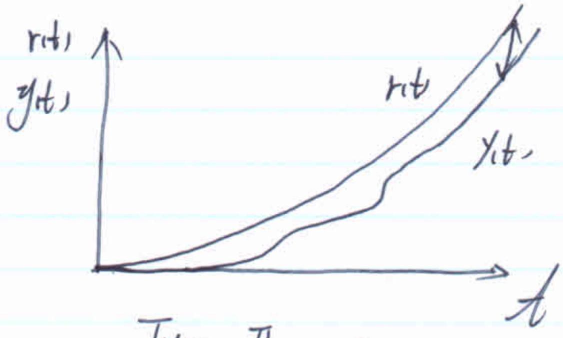
$$e_{ss} = \lim_{s\to0} s \cdot \frac{1}{1+G(s)} \frac{1}{s^3} = \lim_{s\to0} \frac{1}{s^2 G(s)}$$

Define K_a as steady acceleration error coeff
/ acceleration constant

$$K_a = \lim_{s\to0} s^2 G(s) = \begin{cases} 0 & v=0 \\ 0 & v=1 \\ K & v=2 \end{cases}$$

$$\Rightarrow \quad e_{ss} = \begin{cases} \infty & v=0 \\ \infty & v=1 \\ \frac{1}{K_a} & v=2. \end{cases}$$

- Type 0 & I cannot track acceleration signal
- Type II can track, with steady state error. inverse proportional to open-loop gain K.



Type II, acc

Type \ Input	Unit step $r(t)=1(t)$	ramp $r(t)=t$	acceleration $r(t)=\frac{1}{2}t^2$
0	$\frac{1}{1+K_p}$	∞	∞
I	0	$\frac{1}{K_v}$	∞
II	0.	0	$\frac{1}{K_a}$

- K_p. K_v. K_a represent the ability of different types of system when tracking different input signal.

- Type 0 under $1(t)$, Type I under t, Type II under $\frac{1}{2}t^2$, e_{ss} is a constant, open-loop gain ↑ e_{ss} ↓.

- But, cannot raise K to ∞. have to keep system stable

- Type I & II under $1(t)$ $e_{ss}=0$. Thus, Design a control system to be Type I & II can eliminate e_{ss}. However, adding $\frac{1}{s}$ into system may destroy the stability.

- Increase K, add $\frac{1}{s}$. ~~→~~ do please don't affect stability

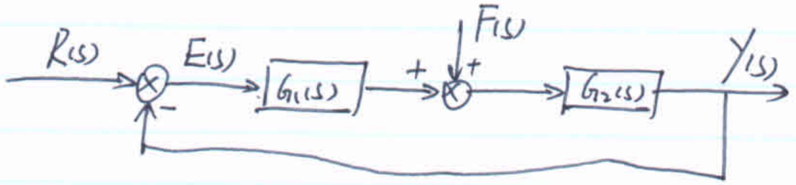
If $r(t) = R_0 \cdot 1(t) + R_1 t + \frac{1}{2}R_2 t^2$.

$$\xrightarrow{\text{Superposition}} e_{ss} = \frac{R_0}{1+K_p} + \frac{R_1}{K_v} + \frac{R_2}{K_a}$$

$\Rightarrow$ e_{ss} depends on system structure & parameters, and input

3) e_{ss} under Disturbance

$$\begin{matrix} r \rightarrow e_{ssr} \\ f \rightarrow e_{ssf} \end{matrix} \xrightarrow{\text{Superposition}} e_{ss} = e_{ssr} + e_{ssf}.$$



Disturbance $F(s)$

Input $R(s)=0$.

So the error signal e(t) is the error of the system under F(s)

As long as we get the error signal $e_f(t)$, and its steady state value, we have e_{ss} of the system under F(s)

Let $R(s) = 0$.

$$E_f(s) = \Phi_{fe}(s) \cdot F(s) = \frac{-G_2(s)}{1+G_1G_2} \cdot F(s)$$

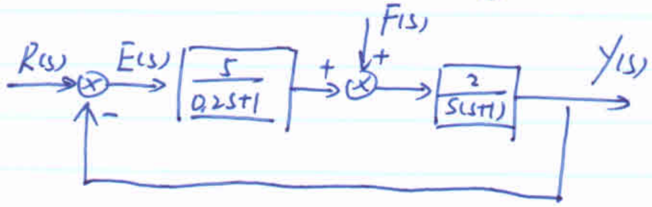
For different f(t), like step, ramp, use FVT get the steady state value of $e_f(t)$

$$e_{sf} = \lim_{t\to\infty} e_f(t) = \lim_{s\to 0} E_f(s) = \lim_{s\to 0} \Phi_{fe}(s) \cdot F(s)$$

Example: Control system as shown in the figure.
Two inputs: $r(t) = t$, $t \geq 0$
$f(t) = 1(t)$, $t \geq 0$

Solve for the steady state error of this system

Solution: Type I system



~~G(s)~~ =

$$G(s) = \frac{5}{0.2s+1} \cdot \frac{2}{s(s+1)} = \frac{10}{s(0.2s+1)(s+1)}$$

characteristic equation: $s(0.2s+1)(s+1) + 10$

$= s(0.2s^2 + 1.2s + 1) + 10$ Routh → Stable

① $r(t) = t \quad f(t) = 0$

$$e_{ssr} = \frac{1}{K_v} = \frac{1}{10} = 0,1$$

② $r(t) = 0 \quad f(t) = 1$

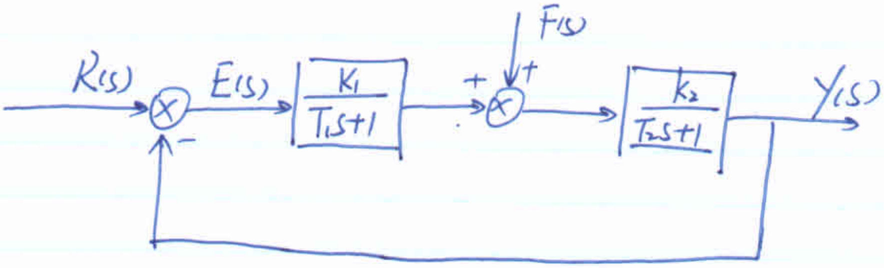
$$e_{ssf} = \lim_{s \to 0} s\, \bar{\Phi}_{fe}(s) \cdot F(s)$$

$$= \lim_{s \to 0} s \frac{-2(0,2s+1)}{s(s+1)(0,2s+1)+10} \cdot \frac{1}{s} = -0,2.$$

$$e_{ss} = e_{ssr} + e_{ssf} = -0,1$$

Example: System as shown in the figure.

(1) $r(t) = 0$. $f(t) = 1(t)$. $\longrightarrow e_{ss}$
(2) $r(t) = 1(t)$. $f(t) = 1(t)$ $\longrightarrow e_{ss}$
(3) How to reduce e_{ss}, how to adjust K_1 & K_2?
(4) Add $\frac{1}{s}$ before &/ after disturbance, what's the difference?



Solution: Type 0 system. Can be easily proved that as long as $K_1 > 0$, $K_2 > 0$, $T_1 > 0$, $T_2 > 0$, closed-loop system is stable

Solution:

No feedforward $\to v=1$

$$\bar{\Phi}_e(s) = \frac{E(s)}{R(s)} = \frac{\textcircled{s}(T_1s+1)(T_2s+1)}{s(T_1s+1)(T_2s+1)+K_1K_2}$$

Add feedforward.

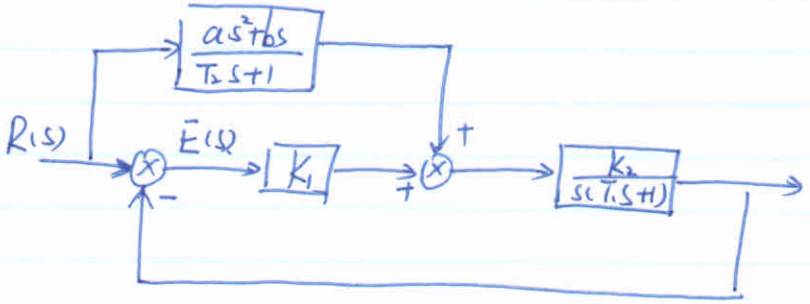
$$\bar{\Phi}_e(s) = \frac{E(s)}{R(s)} = \frac{s(T_1s+1)(T_2s+1) - G_b(s)K_2(T_1s+1)}{s(T_1s+1)(T_2s+1)+K_1K_2}$$

$$= \frac{T_1T_2s^3+(T_1+T_2)s^2+s-G_b(s)K_2(T_1s+1)}{s(T_1s+1)(T_2s+1)+K_1K_2}$$

If let the coefficients of s & $s^0 = 0$, there will be. s^2 factor. $v \to 2$. So we take $G_b(s) = \lambda_1 s$ $\lambda_1 = 1/k_2$.

$$\Rightarrow \bar{\Phi}_e(s) = \frac{E(s)}{R(s)} = \frac{T_1T_2s^3+T_2s^2}{s(T_1s+1)(T_2s+1)+K_1K_2} \qquad v=2.$$

Example:



Requirement: $r(t) = \frac{1}{2}t^2$. $e_{ssr} = 0$ a. b ?

Solution: $r(t) = \frac{1}{2}t^2$ $R(s) = \frac{1}{s^3}$. If $e_{ssr} = 0$. $\bar{\Phi}_e(s)$ must have s^3 in the numerator. $\to v=3$

$$\Phi_e(s) = \frac{E(s)}{R(s)} = \frac{s(T_1s+1)(T_2s+1) - k_2(as^2+bs)}{s(T_1s+1)(T_2s+1)+k_1k_2(T_2s+1)}$$

$$= \frac{T_1T_2s^3+(T_1+T_2-k_2a)s^2+(1-k_2b)s}{s(T_1s+1)(T_2s+1)+k_1k_2(T_2s+1)}$$

To make $v=3$. Coefficients of s^2. s, should be 0.

$$\begin{cases} T_1+T_2-k_2a=0. \\ 1-k_2b=0. \end{cases}$$

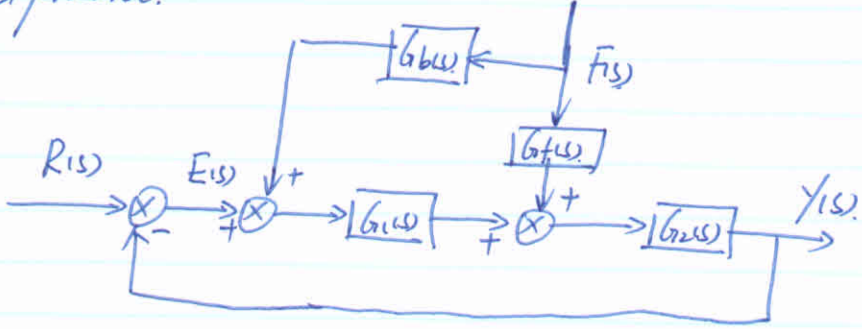
$$\Rightarrow \begin{cases} a=\dfrac{T_1+T_2}{k_2}. \\ b= 1/k_2. \end{cases}$$

* Feedforward only introduces an additional input signal. so it doesn't affect the stability of the closed-loop system.

Reason of using feedfor: If adding $\frac{1}{s}$ in the system to make $v=3$. & keep the stability, too difficult.

2) Reduce / Eliminate e_{ssf}.

* If ~~we ca~~ the disturbance is detectable., then we can use feedforward.



Feedforward compensation: Disturbance applies to the system through the feedforward channel to to cancel the effect on the output caused by the disturbance.

No feedforward

$$Y_f(s) = \frac{G_f(s)\,G_2(s)}{1+G_1(s)G_2(s)}\,F(s)$$

Feedforward:

$$Y_f(s) = \frac{G_f(s)\,G_2(s)}{1+G_1(s)G_2(s)}\cdot F(s) + \frac{G_b(s)\,G_1(s)\,G_2(s)}{1+G_1(s)G_2(s)}\,F(s)$$

$$= \frac{G_f(s)G_2(s) + G_b(s)\,G_1(s)\,G_2(s)}{1+G_1(s)\,G_2(s)}\,F(s)$$

If we can take $G_b(s) = -\frac{G_f(s)}{G_1(s)}$, $F(s)=0$.
No effect at all.

Example: Above figure. $G_f(s) = K_f$, $G_1(s) = \frac{K}{T_1 s+1}$.
$G_2(s) = \frac{k_2}{s(T_2 s+1)}$, Solve for $G_b(s)$ which makes
the effect of $F(s)$ is 0.

Solution: Output caused by $F(s)$ is

$$Y_f(s) = \frac{G_2(s)\left[G_f(s) + G_b(s)\,G_1(s)\right]}{1+G_1(s)\,G_2(s)}\,F(s)$$

take $G_b(s) = -G_f(s)/G_1(s) = -K_f/k_1\,(T_1 s+1)$

Conclusion:

△ Using feedforward can eliminate/reduce ~~e~~ both of e_{ssr} & e_{ssf}.

△ Advantage: Doesn't affect the stability of the closed-loop system

△ Disadvantage: High-order derivatives of Input/Disturbance may be used. Hard to realize in practical applications.

(1) $\Phi_{fe}(s)=\frac{E(s)}{F(s)}=\frac{-K_2(T_1s+1)}{(T_1s+1)(T_2s+1)+K_1K_2}$

FVT

$$e_{ssf}=\lim_{s\to 0} s\cdot\Phi_{fe}(s)\cdot F(s)=\lim_{s\to 0} s\cdot\frac{-K_2(T_1s+1)}{(T_1s+1)(T_2s+1)+K_1K_2}\frac{1}{s}$$

$$=\frac{-K_2}{1+K_1K_2}$$

(2) $e_{ssr}=\frac{1}{1+K_p}=\frac{1}{1+K_1K_2}$

$$e_{ss}=e_{ssr}+e_{ssf}=\frac{1-K_2}{1+K_1K_2}$$

(3) From the expressions of e_{ss}, e_{ssr}, e_{ssf}: ① $K_1\uparrow$ can reduce e_{ssr}, e_{ssf} simultaneously;

② $K_2\uparrow$ can reduce e_{ssr}, ~~not too~~ ~~[illegible]~~ almost nothing to do with e_{ssf}

③ $K_1\uparrow$ $K_2\downarrow$. reduce ~~e_{ssf}~~ e_{ss} and keep the open-loop gain

~~[illegible] → $e_{ss}\downarrow$.~~

(4) ① Add v $\frac{1}{s}$ before disturbance.

$$G_1(s)=\frac{K_1}{s^v(T_1s+1)}$$

$$\Phi_e(s)=\frac{E(s)}{R(s)}=\frac{s^v(T_1s+1)(T_2s+1)}{s^v(T_1s+1)(T_2s+1)+K_1K_2}$$

no error degree $0\rightarrow v$. eliminate e_{ssr}.
System type.

$$\Phi_{fe}(s) = \frac{E(s)}{F(s)} = \frac{-s^{v}K_2(T_1s+1)}{s^{v}(T_1s+1)(T_2s+1)+K_1K_2}$$

no-error degree $0 \rightarrow v$. eliminate e_{ssf}.

②. v $\frac{1}{s}$. after disturbance

$$G_2(s) = \frac{K_2}{s^{v}(T_2s+1)}$$

$$\Phi_e(s) = \frac{E(s)}{R(s)} = \frac{s^{v}(T_1s+1)(T_2s+1)}{s^{v}(T_1s+1)(T_2s+1)+K_1K_2}$$

s^{v} no-error degree $0 \rightarrow v$ eliminate e_{ssr}

$$\Phi_{fe}(s) = \frac{E(s)}{F(s)} = \frac{-K_2(T_1s+1)}{s^{v}(T_1s+1)(T_2s+1)+K_1K_2}$$

no s^{v} in the numerator, no-error degree still 0
cannot eliminate the e_{ssf}.

$\Rightarrow$ Conclusion: 1) Adding $\frac{1}{s}$ before disturbance can ~~reduce~~ eliminate both e_{ssr} & e_{ssf}

2) Adding $\frac{1}{s}$ after disturbance can only ~~reduce~~ eliminate e_{ssr}.

Example: Unit feedback system

$$G(s) = \frac{K}{s(T_1s+1)(T_2s+1)}$$

if $r(t) = a\cdot 1(t) + bt$ $(a>0,\ b>0)$, require $e_{ss} < \varepsilon_0$.
Solve for the conditions of K, T_1, T_2.

Solution: ① Stability

$$D(s) = T_1T_2 s^3 + (T_1+T_2)s^2 + s + K = 0.$$

Routh:

$$\begin{array}{lll} s^3 & T_1T_2 & 1 \\ s^2 & T_1+T_2 & K. \\ s^1 & \dfrac{T_1+T_2-T_1T_2K}{T_1+T_2} & 0 \\ s^0 & K. & \end{array}$$

$$\Rightarrow \begin{cases} T_1 > 0. \\ T_2 > 0 \\ 0 < K < \dfrac{T_1+T_2}{T_1T_2}. \end{cases}$$

② Type I system.

$$r(t) = \underbrace{a \times 1(t)}_{\substack{\downarrow \\ e_{ssr_1}=0.}} + \underbrace{bt}_{\substack{\downarrow \\ e_{ssr_2} = \frac{b}{K_v} = \frac{b}{K}.}}$$

$$\frac{b}{K_v} < \varepsilon_0. \Rightarrow k > b/\varepsilon_0$$

$$\Rightarrow \begin{cases} T_1 > 0 \\ T_2 > 0 \\ \dfrac{b}{\varepsilon_0} < K < \dfrac{T_1+T_2}{T_1T_2} \end{cases}$$

5. Methods to reduce e_{ss}

⇒ ① Increase Open-Loop Gain.

1) Reduce e_{ssr}

For Type 0, I, & II systems, Open-Loop Gain corresponds to K_p, K_v & K_a.

From the Table,

- $K\uparrow$, e_{ssr} of Type 0 under $1(t)$ can be reduced
- $K\uparrow$, e_{ssr} of Type I under t can be reduced.
- $K\uparrow$, e_{ssr} of Type II under $\frac{1}{2}t^2$ can be reduced.

Note that: $K\uparrow$ can only reduce the error under a specific input signal. For $e_{ssr}=0/\infty$, $K\uparrow$ does nothing. It means $K\uparrow$ cannot change the property of the error. However, $K\uparrow$ can reduce the speed of e_{ssr} goes to ∞.

Example: Unit feedback system, open-loop TF as below

$$G(s) = \frac{K}{(0.1s+1)(0.5s+1)}$$

Discuss $K=10$ & $K=100$, e_{ssr} of $r(t)=1(t)$, t & t^2

Solution: $K>-1$ System is stable. Type 0 system

$r(t)=t$, $r(t)=\frac{1}{2}t^2$, $e_{ssr}=\infty$.

$K=10.$

$$\Phi(s)=\frac{E(s)}{R(s)}=\frac{(0.1s+1)(0.5s+1)}{(0.5s+1)(0.1s+1)+10}=\frac{1+0.6s+0.05s^2}{11+0.6s+0.05s^2}$$

$$e_{ssr}=\frac{1}{11}$$

$K=100$

$$\Phi(s)=\frac{E(s)}{R(s)}=\frac{(0.1s+1)(0.5s+1)}{(0.5s+1)(0.1s+1)+100}=\frac{1+0.6s+0.05s^2.}{101+0.6s+0.05s^2}$$

$$e_{ssr}=\frac{1}{101}$$

Conclusion: K↑. can reduce the steady state error ~~of~~ under a specific input signal. But cannot change the property of the error.

2) Reduce e_{ssf}.

We, have, seen before. that. Increase the gain, K_1 which is before the disturbance can, reduce e_{ssf}. while increasing K_2, which is after disturbance doesn't work.

⇒. If we decide to ~~use~~ reduce the e_{ssf} by ~~using a~~ increasing the open-loop gain. please be careful about the place of the gain. If, we can K_1↑ K_2↓. Then the total gain will stay the same, but e_{ssf} ↓.

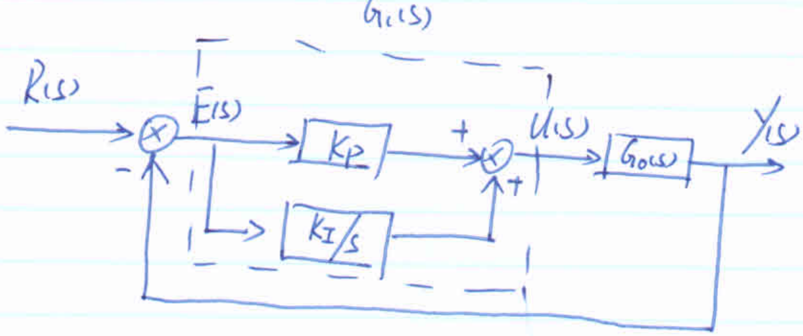
⇒ Increase K properly. otherwise. system will be unstable.

⇒ ② Adding integral

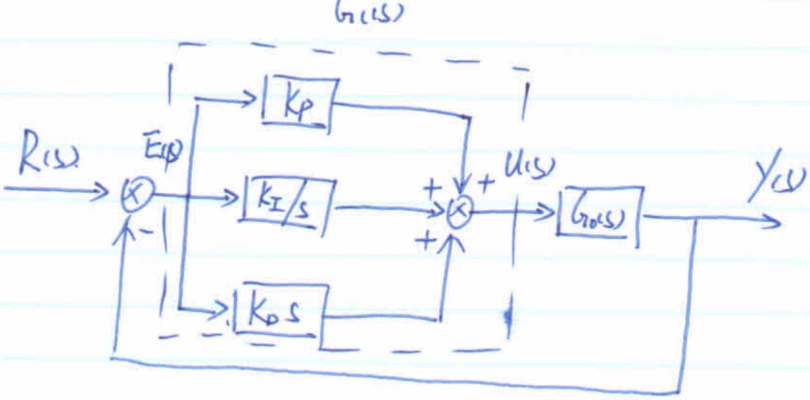
Adding $\frac{1}{s}$ can increase the type of the system, this can change the type of the error.

1) Reduce / Eliminate e_{ssr}.

Using PI / PID is a common method of adding $\frac{1}{s}$ in the system



PI



PI controller: $G_1(s) = K_p + K_I/s = \frac{K_p s + K_I}{s}$ ——— $\frac{1}{s}$.

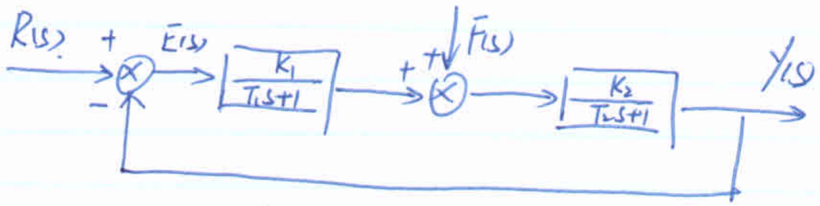
PID controller: $G_1(s) = K_p + K_I/s + K_D s = \frac{K_D s^2 + K_p s + K_I}{s}$

$\infty \rightarrow$ constant   constant $\rightarrow 0$.

2) Reduce / Eliminate e_{ssf}

Adding $\frac{1}{s}$ before disturbance, can increase the type
after. doesn't work.

⇒ Conclusion: By adding $\frac{1}{s}$ ~~we add a pole at 0.~~
~~which will~~ may affect the stability.



$$\Phi_{fe}(s)= \frac{\frac{-K_2}{T_2S+1}}{1+\frac{K_1}{T_1S+1}\frac{K_2}{T_2S+1}}$$

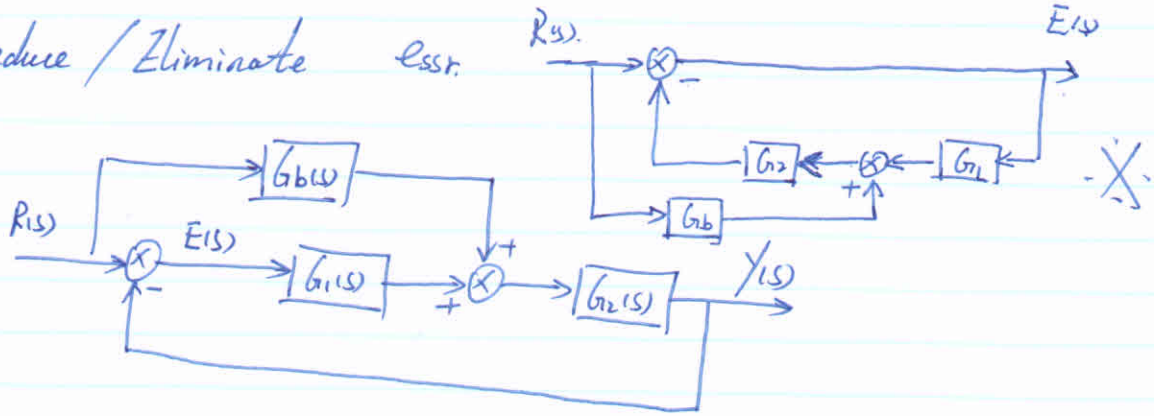
Before: $$\Phi_{fe}(s) = \frac{\frac{-K_2}{T_2S+1}}{1+\frac{K_1}{S(T_1S+1)}\frac{K_2}{T_2S+1}} = \frac{-S(T_1S+1)K_2}{S(T_1S+1)(T_2S+1)+K_1K_2}$$

After: $$\Phi_{fe}(s) = \frac{\frac{-K_2}{S(T_2S+1)}}{1+\frac{K_1}{\cancel{S}T_1S+1}\frac{K_2}{S(T_2S+1)}} = \frac{-K_2(T_1S+1)}{S(T_1S+1)(T_2S+1)+K_1K_2}$$

⇒ ② Feedforward Compensation Control

Use another channel to introduce the input / disturbance to the system. Using the idea of compensation to reduce / eliminate the error.

1) Reduce / Eliminate e_{ssr}.



$G_b(s)$: T.F of Feedforward.

$$\Phi_e(s) = \frac{E(s)}{R(s)} = \frac{1 - G_2(s) \cdot G_b(s)}{1 + G_1(s)\, G_2(s)}$$

If we take $G_b(s) = \frac{1}{G_2(s)}$ $\Rightarrow$ $E(s) = 0 \rightarrow e(t) = 0$.

But it's difficult to have $G_b(s) = \frac{1}{G_2(s)}$, order of numerator will be greater than denominator.

If we take $G_b(s)$ in the form of 1st-order / 2nd-order derivative, the type can be increased, which means e_{ss} can be eliminated / reduced.

Example: Above figure.

$$G_1(s) = \frac{K_1}{T_1 S + 1} \qquad G_2(s) = \frac{K_2}{s(T_2 S + 1)}.$$

If no feedforward, Type I.
If add feedforward. how to choose $G_b(s)$. to increase from Type I to II?

§ Frequency Response.

- A very important technique used in Control System Design.
- As mentioned before, sinusoidal signal is a typical input signal.
- sin(*) & cos(*) have infinite order of derivatives, thus they contain the information of position, velocity, acca, etc.
- A signal can be decomposed to a combination of a group of sinusoidal signals of different frequencies

⇒ By analyzing the system's response to different frequencies ~~of~~ sinusoidal signals, we can have a comprehensive understanding of the system's performance.

Pros: (1) Mathematic model is the frequency model. Physical meaning is distinct.

(2) For some devices ~~the~~ whose mathematical models are hard to derive, we can identify the frequency model using experiment results

(3) Calculation is simple.

Cons: (1) Can only be applied to LTI system

(2) A lot of approximation will be used, the result is not accurate.

Three parts in this chapter

(1) Frequency property.
(2) Using frequency properties / response to analyze the system performance, including stability, dynamic process and error calculation.
(3) Using frequency properties / response to design controller to improve system performance.

§ Frequency Property

§. Steady output ~~a~~ under sinusoidal signal.

LTI system. input $x(t)$
output $y(t)$

$$\text{T.F} \quad G(s) = \frac{Y(s)}{X(s)}$$

$$G(s) = \frac{B(s)}{A(s)} = \frac{B(s)}{(s-s_1)(s-s_2)\cdots(s-s_n)}$$

$$A(s) = s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0 = (s-s_1)(s-s_2)\cdots(s-s_n)$$
$$B(s) = b_m s^m + b_{m-1}s^{m-1} + \cdots + b_1 s + b_0$$

$s_1, s_2 \cdots s_n$ are poles of this system, for a stable system, real parts of $s_1 \cdots s_n$ should be negative.

Input : $x(t) = X \sin \omega t$

X: ~~amplitude~~ ~~Magnitude~~ Amplitude.
ω: angular frequency.

$$X(s) = \frac{X\omega}{(s+j\omega)(s-j\omega)}$$

$$Y(s) = G(s)\,X(s) = \frac{B(s)}{(s-s_1)(s-s_2)\cdots(s-s_n)} \cdot \frac{X\omega}{(s+j\omega)(s-j\omega)}$$

$$Y(s) = \frac{d_1}{s+j\omega} + \frac{d_2}{s-j\omega} + \frac{c_1}{s-s_1} + \frac{c_2}{s-s_2} + \cdots + \frac{c_n}{s-s_n}$$

$c_1 \sim c_n$, d_1, d_2 can be calculated

$$\mathcal{L}^{-1}\{Y(s)\} = d_1 e^{-j\omega t} + d_2 e^{j\omega t} + c_1 e^{s_1 t} + c_2 e^{s_2 t} + \cdots + c_n e^{s_n t}, \quad t \geq 0$$

Since system is stable, $c_1 e^{s_1 t} + \cdots + c_n e^{s_n t} \xrightarrow{t\to\infty} 0$

steady output $y_{ss}(t) = d_1 e^{-j\omega t} + d_2 e^{j\omega t}$

$Y(s)(s+j\omega)\big|_{s=-j\omega}$

$$d_1 = G(s)\frac{X\omega}{(s+j\omega)(s-j\omega)}\cdot(s+j\omega)\Big|_{s=-j\omega} = -\frac{G(-j\omega)X}{2j}$$

$$d_2 = G(s)\frac{X\omega}{(s+j\omega)(s-j\omega)}\cdot(s-j\omega)\Big|_{s=j\omega} = \frac{G(j\omega)X}{2j}$$

$G(j\omega)$ is a complex number, write it in ~~argument~~ ~~amplitude-phase~~ Magnitude-argument

$$G(j\omega) = |G(j\omega)|\, e^{j\phi}$$

$$\varphi = \angle G(j\omega) = \arctan\left[\frac{\mathrm{Im}\,G(j\omega)}{\mathrm{Re}\,G(j\omega)}\right]$$

$$G(-j\omega) = |G(-j\omega)|\, e^{-j\varphi} = |G(j\omega)|\, e^{-j\varphi}$$

$$y_{ss}(t) = X|G(j\omega)|\,\frac{e^{j(\omega t+\varphi)} - e^{-j(\omega t+\varphi)}}{2j}$$

$$= X|G(j\omega)|\sin(\omega t+\varphi)$$

$$= Y\sin(\omega t+\varphi) \quad \text{with} \quad Y = X|G(j\omega)|$$

$\Rightarrow$ For a stable, LTI system, steady output of ~~a~~ this system under a sinusoidal signal is a sinusoidal signal with the same angular frequency as the input signal.

⇒ Amplitude: $y_{ss}(t) : X\,|G(j\omega)|$
$x(t) : X$

⇒ Argument: $y_{ss}(t): \omega t + \varphi$
$x(t): \omega t.$

function of frequency ω.

§ Frequency Response

- $G(j\omega)$ can reflect the relationship between the steady output and input signal (~~different~~ sinusoidal signal with different frequency).

- From $G(j\omega)$, we can directly obtain the steady output

$$y_{ss}(t) = Y \sin(\omega t + \varphi)$$

$$Y = |G(j\omega)|\,X$$

$$\varphi = \angle G(j\omega)$$

- $G(j\omega)$ can reflect the properties of a device/component/system, we define $G(j\omega)$ is the frequency property of the system.

- $G(j\omega)$ is a complex variable, can be obtained directly from transfer function $G(s)$

$$G(j\omega) = G(s)\big|_{s=j\omega} = |G(j\omega)|\, e^{j\varphi(\omega)}$$

Properties:

(1) Frequency property is a mathematic model used to describe LTI system. Only LTI system has frequency property.

(2) $G(j\omega)$ is a complex variable. ~~amplitude~~ Magnitude $|G(j\omega)|$ & phase $\varphi(j\omega)$ are functions of frequency ω.

(3) Input ω sinusoidal function.

$$|G(j\omega)| = \frac{\text{Amp of steady output}}{\text{Amp of input signal}}$$

$$\angle G(j\omega) = (\text{phase of steady output} - \text{phase of input})$$

(4) Unit for ω : rad/s / 1/s.

(5) The range for ω, $\omega \in (0, +\infty)$.
when $\omega < 0$.

$$|G(j\omega)| = |G(-j\omega)|$$

$$\angle G(j\omega) = -\angle G(-j\omega)$$

OR.

$$Re[G(j\omega)] = Re[G(-j\omega)]$$

$$Im[G(j\omega)] = -Im[G(-j\omega)].$$

(6) Another definition for frequency property. The fraction of the Fourier Transforms of output & input.

$$G(s)\big|_{s=j\omega} = \frac{Y(j\omega)}{X(j\omega)} = G(j\omega)$$

(7) Frequency property is the Fourier Transform of the system impulse response.

Input $x(t) = \delta(t)$.

$$\mathcal{L}\{x(t)\} = 1$$

$$\mathcal{F}\{x(t)\} = 1$$

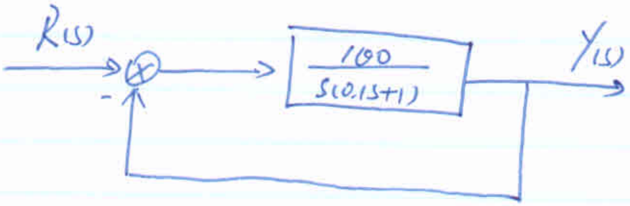
if the impulse response is $k(t)$, we have.

$$G(j\omega) = Y(j\omega) = \int_{-\infty}^{+\infty} k(t)\, e^{j\omega t}\, dt.$$

The method to test the frequency property of the system:

① input is impulse / ~~approximately~~ almost impulse.
② Record $k(t)$
③ $\mathcal{F}$

Example: Block Diagram.



If the input is $r(t) = \sin 5t$, please solve for the steady state error.

Solution: $\Phi_e(s) = \frac{E(s)}{R(s)} = \frac{s(0.1s+1)}{0.1s^2+s+100}$

Input :

Amp	1
Frequency	5
phase	0

steady state error is the steady output of $e(t)$. It is of the same frequency with the input signal.

$$E_0 = |\Phi_e(j\omega)|_{\omega=5} \cdot 1 = 0.057$$

$$\angle \Phi_e(j\omega)|_{\omega=5} = 113.62°$$ ~~116° - 2.~~93° 116.56-2.94

Thus, the steady state error is

$$e_{ss}(t) = E_0 \sin(5t + \angle\Phi_e(j5)) = 0.057 \sin(5t + 113.6°)$$

§ Several forms of the frequency property.

- $G(j\omega)$ is a complex function/variable, it is a function of ω. For a specific ω, $G(j\omega)$ is a point on the complex plane; For continuously changing ω, $G(j\omega)$ represents a trajectory on the complex plane.

① Polar Coordinates (Nyquist plot) *

② Logarithmic Coordinates (Bode plot) *

③ Logarithmic ~~Amp~~ Mag-Pha plot (Nichols plot)

Bode plot:

(1) Separate ~~Amplitude~~ Magnitude property & Phase property.

x-axis : ω

y-axis : ~~Amplitude~~ Magnitude two plots
Phase.

(2) The range for ω is large $0 \to +\infty$.
If we use linear coordinate, x-axis will be too long to see the low/high frequency. So we choose $\log_{10}$ / $\lg \omega$ as the x-axis.
For example, ω_1 & $\omega_2 = 10\omega_1$, the distance between them $\mu = \lg\omega_2 - \lg\omega_1 = \lg 10\omega_1 - \lg\omega_1$

$$= \lg 10 + \lg\omega_1 - \lg\omega_1$$

$$= \lg 10 = 1.$$

By doing so, the unit on x axis is "dec" / decade.

Distance between two different frequency

$\mu = \lg\omega_2 - \lg\omega_1$, unit decade.

(3) ~~Amplitude~~ Magnitude plot: y axis is not $|G(j\omega)|$, but $L(\omega)$.

$L(\omega) = 20\lg|G(j\omega)|$. unit dB.

Pros of using $L(\omega)$: 1) When calculating $K \cdot G(j\omega)$, we have

$$20\lg|KG(j\omega)| = 20\lg K|G(j\omega)| = 20\lg|G(j\omega)| + 20\lg K.$$

The amplitude only needs to move up/down $20\lg K$

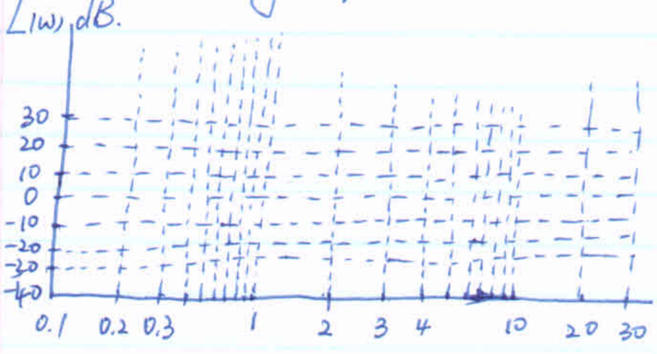
2) Calculating the product of two amplitude ~~p~~ properties: $|G_1(j\omega)| \cdot |G_2(j\omega)|$

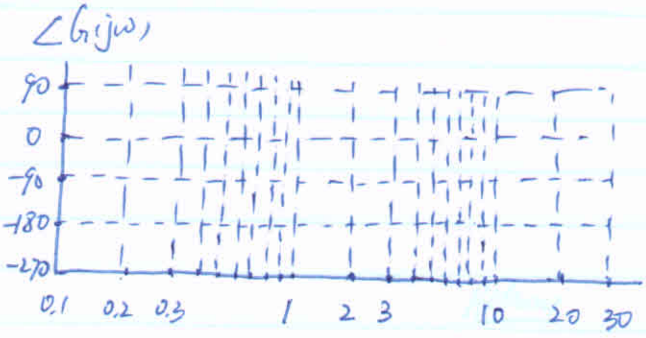
$$20\lg|G_1(j\omega)| \cdot |G_2(j\omega)| = 20\lg|G_1(j\omega)| + 20\lg|G_2(j\omega)|$$

Just need to add them together.

(4) Phase plot: Take the value $\angle G(j\omega)$ itself.

(5) Only plot ω that we are intrested in.





Nyquist Plot:

Write $G(j\omega)$ in Magnitude-~~Argument~~ Phase form, when angular frequency changes from 0° → +∞°, plot the trajectory of $G(j\omega)$ in Complex plane.

Example: Open-loop transfer function

$$G(s) = \frac{1}{TS+1}$$

Plot Nyquist Plot.

Solution:

$$G(j\omega) = G(s)\Big|_{s=j\omega} = \frac{1}{Tj\omega+1}$$

$$|G(j\omega)| = \left|\frac{1}{Tj\omega+1}\right| = \frac{1}{\sqrt{1+(\omega T)^2}}$$

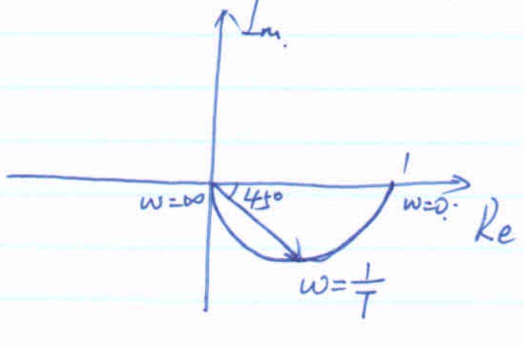
$$\varphi(\omega) = \angle G(j\omega) = \angle \frac{1}{j\omega T+1} = -\arctan \omega T$$

when ω changes from $0 \to +\infty$, $|G(j\omega)|: 1 \to 0$

$\angle G(j\omega): 0 \to -90°$

$\omega = \frac{1}{T}$ $\quad |G(j\omega)|: \frac{\sqrt{2}}{2}$

$\angle G(j\omega): -45°$



Property: $K\,G(j\omega)$. all magnitude multiplied by K. phase stays the same.

§ Frequency response of basic elements

1. Pure Gain

T.F $G(s) = K.$

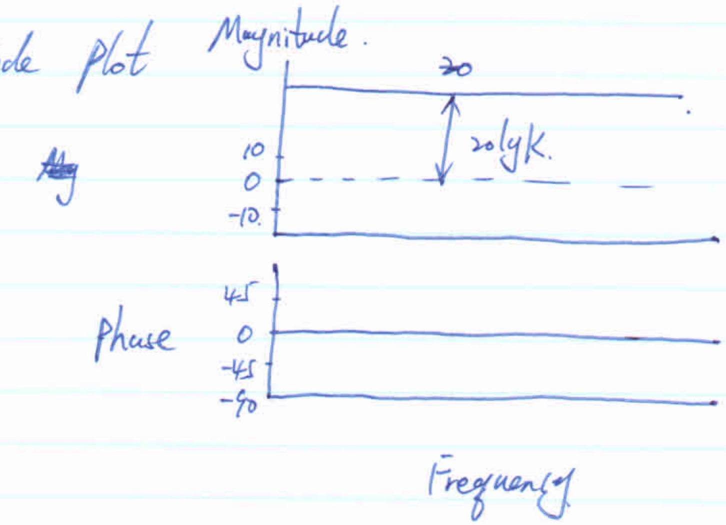
F.R $G(j\omega) = K.$

$$|G(j\omega)| = K.$$

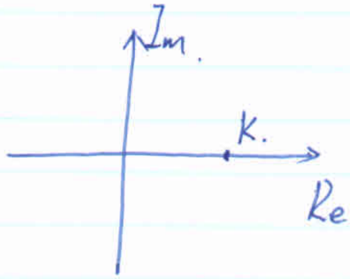
$$\angle G(j\omega) = 0$$

$$L(\omega) = 20\lg|G(j\omega)| = 20\lg K.$$

Bode plot



Nyquist plot



2. Pure integral.

T.F. $G(s) = \frac{1}{s}$

F.R. $G(j\omega) = \frac{1}{j\omega}$

$$|G(j\omega)| = \frac{1}{\omega}.$$

$$\angle G(j\omega) = -90^\circ$$

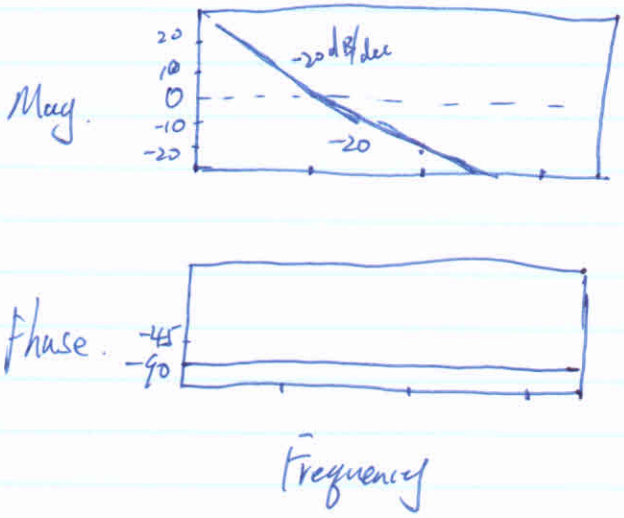
$$L(\omega) = 20\lg|G(j\omega)| = 20\lg\frac{1}{\omega} = -20\lg\omega$$
$$\angle G(j\omega) = -90^\circ$$

$\omega_2 = 10\omega_1$

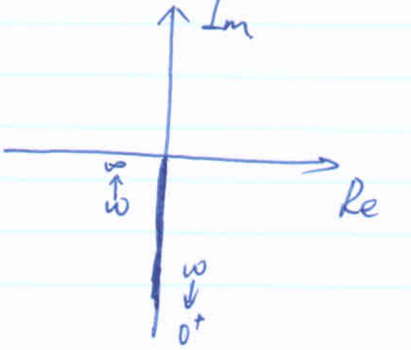
$$L(\omega_2) - L(\omega_1) = -20\lg\omega_2 + 20\lg\omega_1$$
$$= -20\lg 10\omega_1 + 20\lg\omega_1$$
$$= -20 \text{ dB.}$$

- The magnitude will drop 20 dB per decade.
- $\omega = 1$ $L(\omega) = 0$ dB. $L(\omega)$ goes across $\omega = 1$ $L(\omega) = 0$ dB. slope -20 dB/dec.

$\angle G(j\omega)$ line $-90°$.



Nyquist Plot



(3) Inertia element

T.F. $G(s) = \frac{1}{TS+1}$

F.R. $G(j\omega) = \frac{1}{1+j\omega T}$

$$|G(j\omega)| = \frac{1}{\sqrt{\omega^2 T^2+1}}$$

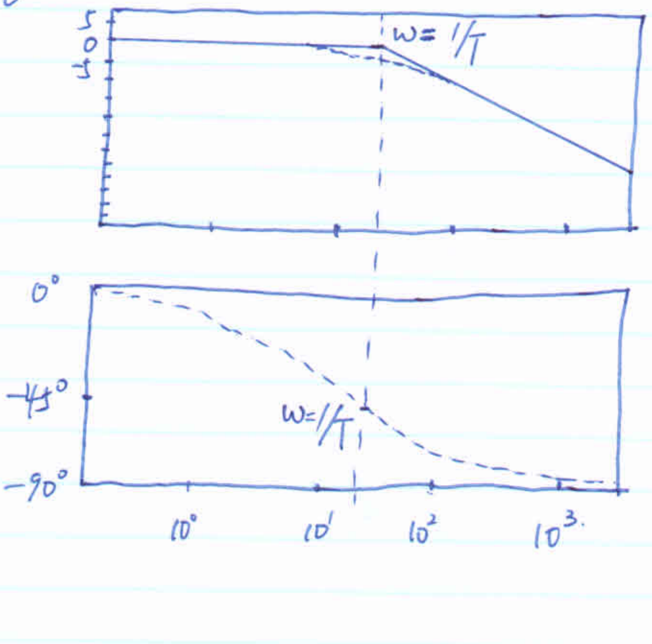
$$\angle G(j\omega) = -\arctan T\omega.$$

- ω - $0 \to +\infty$. Magnitude decays from 1 to 0.
- At $\omega = \frac{1}{T}$. Magnitude $= \frac{1}{\sqrt{2}}$
- ω $0 \to +\infty$ Phase $0° \to -90°$
- At $\omega = \frac{1}{T}$ Phase $-45°$

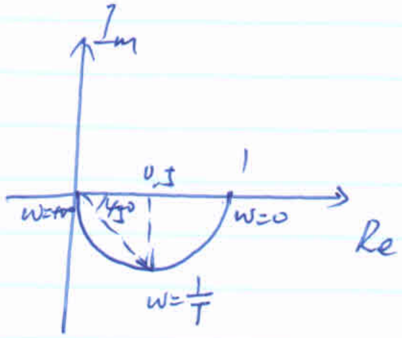
$$L(\omega) = 20\lg|G(j\omega)| = -20\lg\sqrt{T^2\omega^2+1}$$

$$\angle G(j\omega) = -\arctan \omega T$$

Bode Plot



Nyquist Plot



Obviously, the magnitude can be approximated by a Broken-Line. If the we do not requre the accuracy, we can use the broken line to represent.

- In low frequency, which means $\omega << \frac{1}{T}$.

$$L(\omega) = -20\lg\sqrt{T^2\omega^2+1} = 0.$$

- In high frequency, which means $\omega >> \frac{1}{T}$

$$L(\omega) = -20\lg\sqrt{T^2\omega^2+1} = -20\lg T\omega$$

A line going across $\underset{L(\omega)=0}{\omega = \frac{1}{T}}$. slope $-20dB/dec$.

- The above two lines intersects at $\omega = \frac{1}{T}$ $L=0$ dB. The broken-line constructed by the above two lines is the approximated magnitude response of the inertia element.

- $\omega = \frac{1}{T}$ corner frequency

- From the bode plot (magnitude), the maximum error between the actual & approximated magnitude exists at $\omega = \frac{1}{T}$

$$-20\lg\sqrt{1+T^2\omega^2}\Big|_{\omega=\frac{1}{T}} - (-20\lg\omega T)\Big|_{\omega=\frac{1}{T}} = -20\lg\sqrt{2} = -3dB$$

$\omega = 0.1\frac{1}{T} \sim 10\frac{1}{T}$

$\omega/(1/T)$	0.1	0.25	0.4	0.5	1.0	2.0	2.5	4.0	10
error/dB.	-0,04	-0,32	-0,65	-1.0	-3,01	-1,0	-0,65	-0,32	-0,04

- Phase plot (a continuous curve from $0° \sim -90°$, $w = \frac{1}{T}$, $-45°$)

$w/(1/T)$	0.1	0.25	0.4	0.5	1.0	2.0	2.5	4.0	10
phase/°	-5.7	-14.1	-21.8	-26.6	-45	-63.4	-68.2	-75.9	-84.3

$w < 0.1\frac{1}{T}$ & $w > 10\frac{1}{T}$. approximated by $0°$ & $90°$.

4) Oscillation element

T.F $$G(s) = \frac{1}{T^2s^2 + 2\xi Ts + 1} \qquad 0 < \xi < 1$$

T: time constant of oscillation element
ξ: damping ratio of oscillation element

Magnitude: $$|G(jw)| = \frac{1}{\sqrt{(1-T^2w^2)^2 + (2\xi wT)^2}}$$

Phase: $$\angle G(jw) = -\arctan\frac{2\xi Tw}{1-T^2w^2}$$

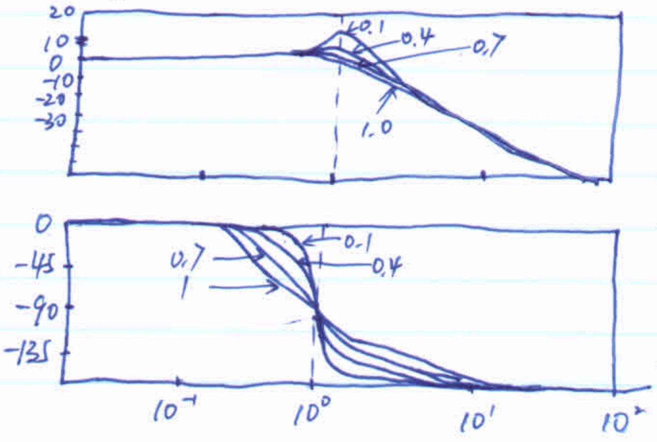
$w=0$ $|G(jw)| = 1$
$\angle G(jw) = 0°$

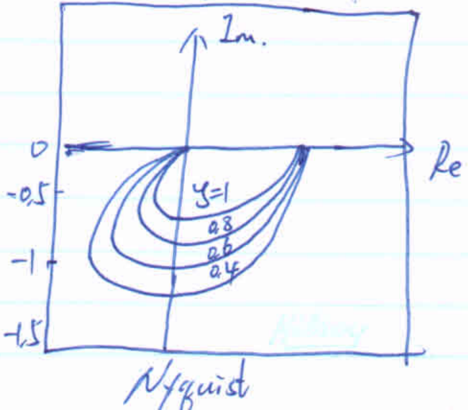
$w = \frac{1}{T}$ $|G(jw)| = \frac{1}{2\xi}$
$\angle G(jw) = -90°$

$$L(w) = 20\lg\sqrt{\frac{1}{(1-T^2w^2)^2 + (2\xi wT)^2}} = -20\lg\sqrt{(1-T^2w^2)^2 + (2\xi Tw)^2}$$

$$\angle G(jw) = -\arctan\frac{2\xi wT}{1-T^2w^2}$$

$w = +\infty$ $|G(jw)| \to 0$
$\angle G(jw) \to -180°$





Nyquist

- It can be obsereved from the bode plot that, when the ζ is around 0.5, and we don't require too much on the accuracy, the magnitude plot can be approximated by ~~the~~ a broken-line.

- $\omega << \frac{1}{T}$, $L(\omega) \approx 0dB$.

- $\omega >> \frac{1}{T}$ $\quad L(\omega)/dB \approx -20lg\sqrt{(T^2\omega^2)^2} = -40lgT\omega$
  This means that the magnitude plot goes across $\omega = 1/T$ $L(\omega)=0$, slope is $-40 dB/dec$.

- The above two lines intersects $\omega = 1/T$ (corner frequency is $\omega = 1/T$).

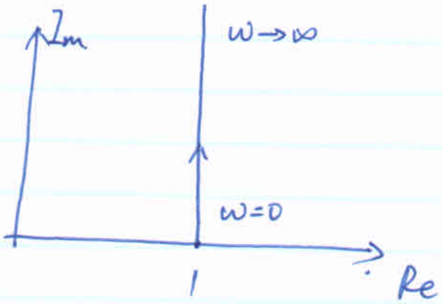
- For different damping ratio ζ, at corner frequency $\omega = \frac{1}{T}$, difference between accurate & ~~app~~ approximation follows

ζ.	0.05	0.1	0.15	0.2	0.25	0.3	0.4	0.5	0.6	0.7	0.8	1.0
Adjust/dB	+20	+40	+10.5	+8	+6	+4.4	+1.94	0	-1.6	-2.92	-4	-6.

5) First order derivative element

$$T.F.: \quad G(s) = \tau s + 1$$

τ: time constant

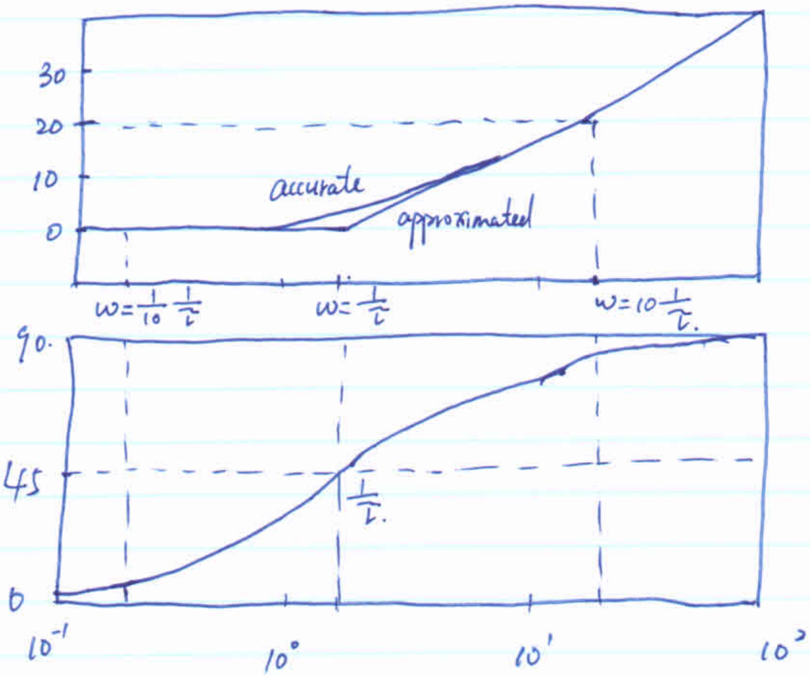


Magnitude: $|G(j\omega)| = \sqrt{1+\tau^2\omega^2}$

Phase: $\angle G(j\omega) = \arctan \tau\omega$.

$$L(\omega)/dB = 20\lg\sqrt{1+\tau^2\omega^2}$$
$$\angle G(\omega) = \arctan \tau\omega$$

Bode Plot:



- When $\omega \ll \frac{1}{\tau}$, $L(\omega) \approx 0$dB.
- When $\omega \gg \frac{1}{\tau}$, $L(\omega) \approx 20\lg \tau\omega$, it intersects $L(\omega)=0$ at $\omega = 1/\tau$. slope = 20 dB/dec.
- The above two lines construct a broken-line at the corner frequency $\omega = 1/\tau$
- ω $0 \to +\infty$. phase $0° \to +90°$
  $\omega = 1/\tau$. phase = 45°
- Magnitude & Phase of first-order derivative element are symmetric to ~~int~~ inertia element.

6) Second order derivative element

$$\text{T.F.} \quad G(s) = \tau^2 s^2 + 2\zeta\tau s + 1$$

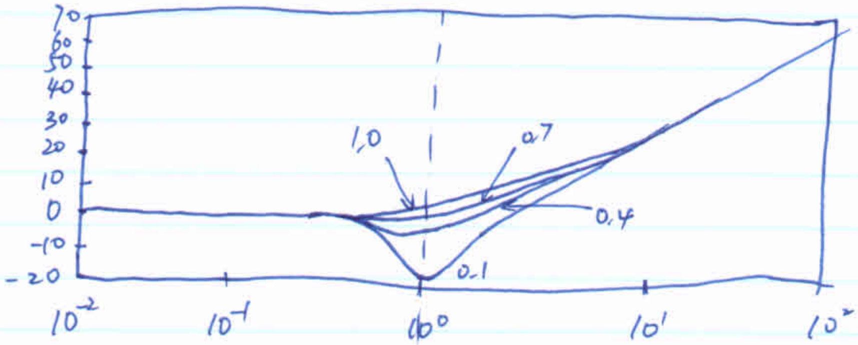
τ: time constant

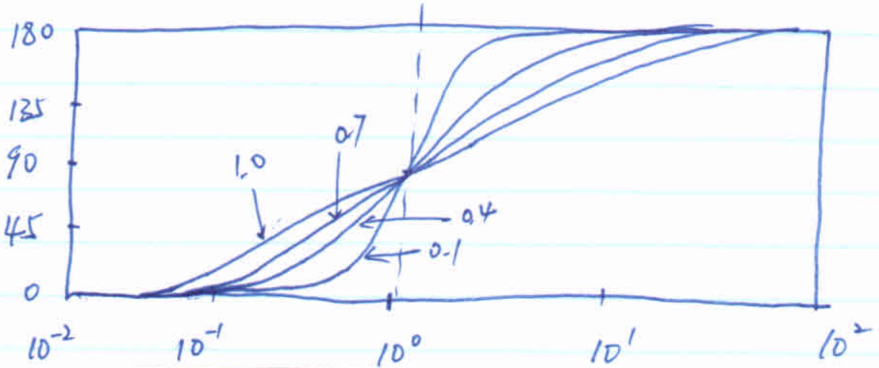
ζ: damping ratio

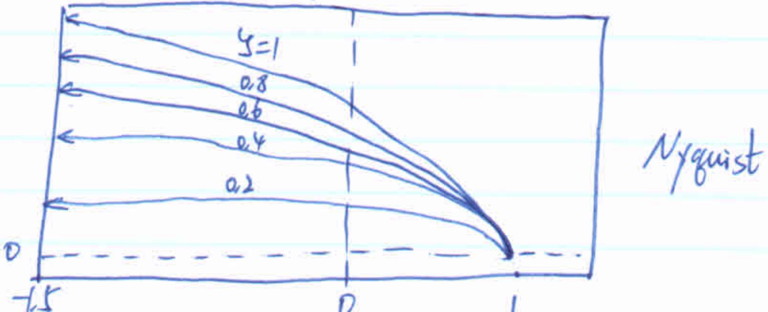
Magnitude : $|G(j\omega)| = \sqrt{(1-\tau^2\omega^2)^2 + (2\zeta\tau\omega)^2}$

Phase : $\angle G(j\omega) = \arctan \frac{2\zeta\omega\tau}{1-\tau^2\omega^2}$

$$L(\omega) = 20\lg\sqrt{(1-\tau^2\omega^2)^2 + (2\zeta\tau\omega)^2}$$
$$\angle G(j\omega) = \arctan \frac{2\zeta\tau\omega}{1-\tau^2\omega^2}$$







- Magnitude plot is a broken line with slope 0 dB/dec & +40dB/dec. $(\omega = \overset{\text{corner}}{\frac{1}{\tau}}.\ L(\omega) = 0dB)$

- $\omega = 0 \to +\infty$. phase $0° \to +180°$, $\omega = 1/\tau$. phase $= +90°$

- Symmetric to oscillation element.

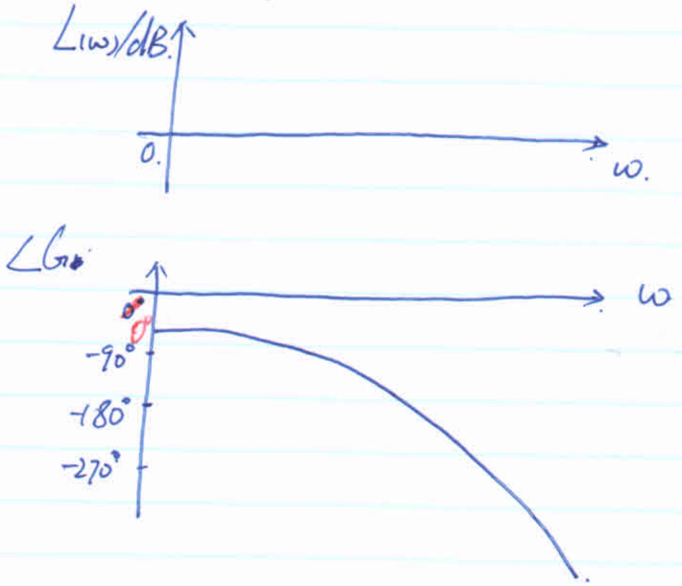
7) Delay element

$$\text{T.F.} \quad G(s) = e^{-\tau s}$$

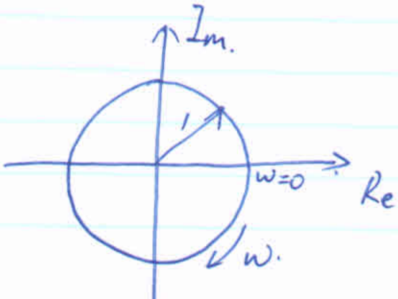
$$|G(j\omega)| = 1.$$

$$\angle G(j\omega) = -\tau\omega$$

Bode



Nyquist



8) Non minimum phase element

In transfer function, if any pole/zero has positive real part, this element is called non-minimum phase element; Otherwise, if all the poles/ & zeros have negative real parts, including imag axis this element is called minimum phase element.

For example: $G(s) = \frac{K}{TS-1}$

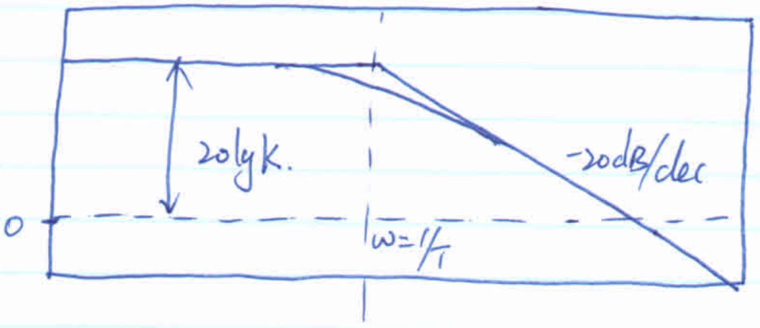
cos − sin +

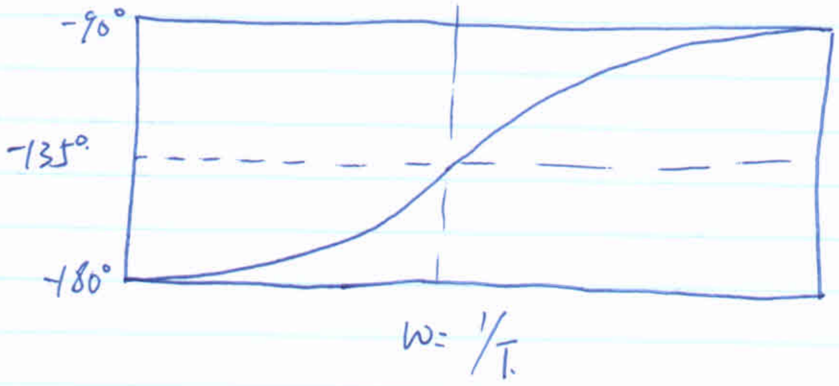
2nd

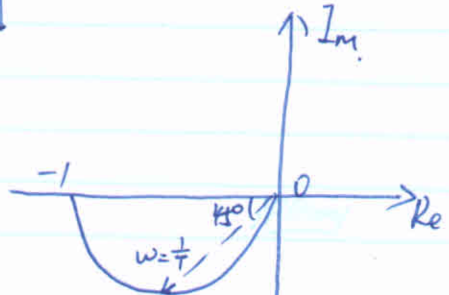
$\frac{1}{\square}$ 3rd.

Magnitude: $|G(j\omega)| = \frac{K}{\sqrt{T^2\omega^2+1}}$

Phase : $\angle G(j\omega) = -\arctan(\frac{\omega T}{-1}) - \pi$.







Note:

(1) Pay attention to the signs of real & imag parts, in order to determine the phase quadrant.

For example: T.F. $G(s) = \frac{1}{TS-1}$

$$\angle G(j\omega) = -\arctan\left(\frac{T\omega}{-1}\right) - \pi.$$

cos ~~ϕ~~ − sin +
2nd $\frac{1}{}$
3rd.

arctan $-\frac{\pi}{2}, \frac{\pi}{2}.$

T.F. $G(s) = \frac{1}{-TS+1.}$

$$\angle G(j\omega) = -\arctan \frac{-T\omega}{1.}$$

(2) Magnitude response of non-minimum phase element is the same as first-order inertia element.

Phase responses are different, shift > minimum phase., phase.

For other elements, if the magnitude responses are the same, phase shift of minimum-phase element is smaller than. non-minimum-phase element.

§ Frequency Response of Control Systems

§ Nyquist plot of Open-loop systems

- Open-loop T.F. can be written as a product of many basic elements.

$$G(s) = \frac{K(\tau s+1)}{s(T_1 s+1)(T_2^2 s^2+2\xi T_2 s+1)}$$

Gain: $G_1(s) = K$

1st-order Derivative: $G_2(s) = \tau s+1$

Integral: $G_3(s) = 1/s$

Inertia: $G_4(s) = \frac{1}{T_1 s+1}$

Oscillation: $G_5(s) = \frac{1}{T_2^2 s^2+2\xi T_2 s+1}$

$$G(j\omega) = G_1(j\omega) \cdot G_2(j\omega) \cdot G_3(j\omega) \cdot G_4(j\omega) \cdot G_5(j\omega)$$

$$|G(j\omega)| = |G_1(j\omega)| * |G_2(j\omega)| * |G_3(j\omega)| * |G_4(j\omega)| * |G_5(j\omega)|$$

$$\angle G(j\omega) = \angle G_1(j\omega) + \angle G_2(j\omega) + \cdots + \angle G_5(j\omega)$$

Magnitude: ~~add~~ multiply together
Phase: add together

Steps to plot Nyquist plot:

① $\omega=0$, Gain, Inertia, Oscillation, 1st-order derivative and 2nd-order derivative Phase $=0°$. Only when $\frac{1}{s}$ exists, at $\omega=0$ phase $= -90°$. Thus, when system has ν $\frac{1}{s}$, Nyquist plot starts from $\omega=0_+$, phase $= -\nu 90°$.

② $\omega = 0$. $|Gain| = K$ $|\frac{1}{s}| = \infty$. $|other\ elements| = 1$.
Thus. if no $\frac{1}{s}$, at $\omega = 0$. magnitude $= K$.
has $\frac{1}{s}$. at $\omega = 0$. magnitude $= \infty$.

③ $\omega \to +\infty$. from the Nyquist plot of every minimum phase basic elements, ~~every~~ 1) tangent of Inertia, Integral and Oscillation is order $\times -90°$, 2) tangent of 1st-order derivative, 2nd-order derivative is order $\times 90°$. Thus, when $\omega \to +\infty$. tangent of Nyquist Plot is $(n-m) \times -90°$, m is the order of numerator. n is the order of denominator. Usually, $n > m$.

④ $\omega \to +\infty$. except for pure Gain, 1st, 2nd. derivative other minimum phase elements have ~~phase~~ magnitude 0. Thus, when $\omega \to +\infty$. open-loop Nyquist Plot approaches origin ~~from the~~ with the direction $(n-m)(-90°)$.
Depends on the order of numerator & denominator.

Example: Open-loop T.F.

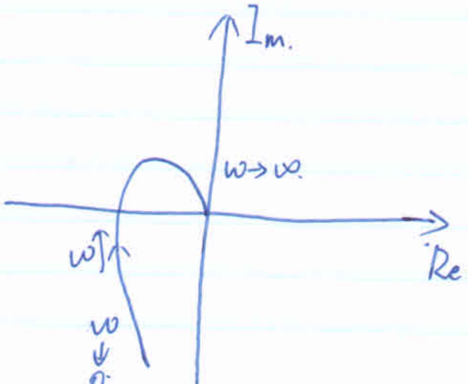
$$G(s) = \frac{2(s+1)}{s(0.5s+1)(0.8s^2+0.9s+1)}$$

plot a draft of Nyquist Plot.

Solution: One integral. $\omega = 0+$, magnitude $= \infty$
phase $= -90°$

$\omega \to +\infty$. $n = 4$. $m = 1$

tangent $(n-m)(-90°) = -270°$
towards origin.



§ Bode Plot of Open-loop Systems

The open-loop T.F. is a cascade of several minimum phase basic elements.

$$20\lg|G(j\omega)| = 20\lg|G_1(j\omega)| + 20\lg|G_2(j\omega)| + \cdots + \cdots$$

$$= L_1(\omega) + L_2(\omega) + \cdots + \cdots$$

Magnitude: Add each lg(Mag) together

Phase: Add each $\phi\angle$ together

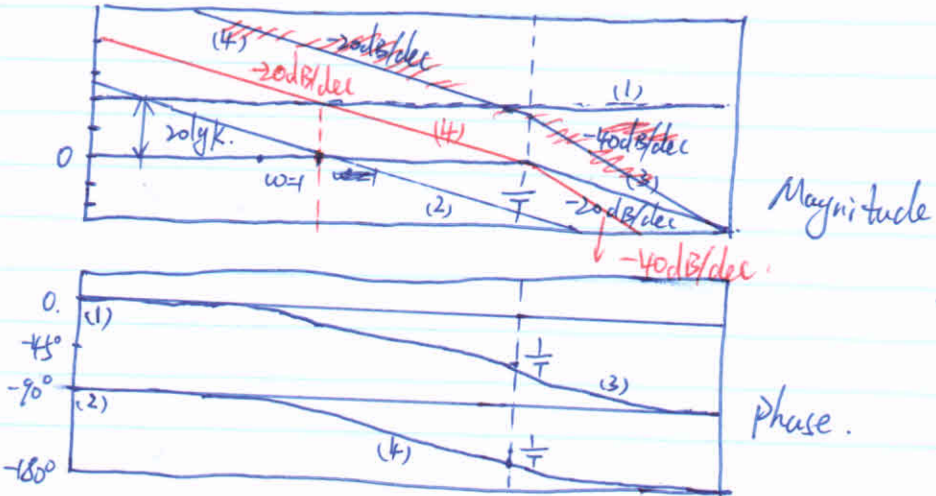
Make Bode plot for each element, then add them together

Example: Open-loop T.F.

$$G(s) = \frac{K}{s(Ts+1)}$$

Plot the draft of Bode Plot

Solution: One Gain (1), One Integral (2), One Inertia (3)

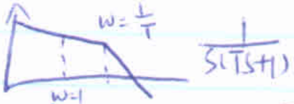


Steps to plot Bode plot

① Write the open-loop T.F. into basic elements form. $(s+T)/(Ts+1)$

② List all time constants of Inertia, Oscillation, 1st-order Derivative, 2nd-order derivative from high to low, ~~each~~ reciprocals of each time constant corresponds to a corner frequency, corner frequencies will be listed from low to high.

③ The very left side (lowest frequency side), only the lg(Mag) of Gain and Integral are not 0, others' are all 0dB. Thus, at this side, it is determined by K/s^{v}.
At lowest frequency side, lg(Mag) goes through $w=1$ $L(w)=20\lg K$.
or its extension
slope $= v \cdot (-20\,dB/dec)$. v is the number of integals.



④ ~~lg(Mag) is a~~ The asymptote of lg(Mag) is a continuous broken line, slope changes at each corner frequency. From left to right, increment of slope depends on the element.

Inertia	-20dB/dec
Oscillation	-40dB/dec
1st-order derivative	+20dB/dec
2nd-order derivative	+40 dB/dec

⑤ At the highest frequency side, slope of lg(Mag) is $(n-m)(-20\,dB/dec)$

⑥ If oscillation element with $\zeta<0.3$/$\zeta>0.8$ exists, some adjustment should be made according to the Table.

⑦ Phase plot is a continuous, smooth and gradually varied curve. At the lowest frequency side, phase = $\nu \cdot (-90°)$.

⑧ Increment of phase, from left to right, at corner frequency depends on elements.

Inertia	$-90°$
Oscillation	$-180°$
1st-order derivative	$+90°$
2nd-order derivative	$+180°$

⑨ Phase Plot, very right side, (highest frequency side) phase = $(n-m)(-90°)$.

⑩ If non-minimum phase element exists, consider it separately.

⑪ If time delay $e^{-\tau s}$ exists, $lg(Mag) = 0\,dB$, only need to consider its phase.

Example: Open-loop T.F.

$$G(s) = \frac{10(s+1)}{s(2.5s+1)(0.04s^2+0.24s+1)}$$

plot the asymptote of lg(Mag) and Phase.

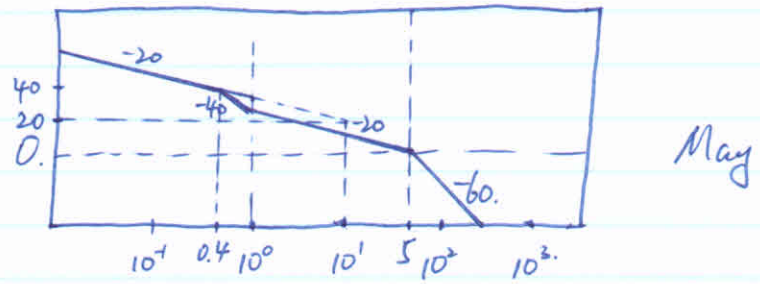
Solution: ① This is a minimum phase system.

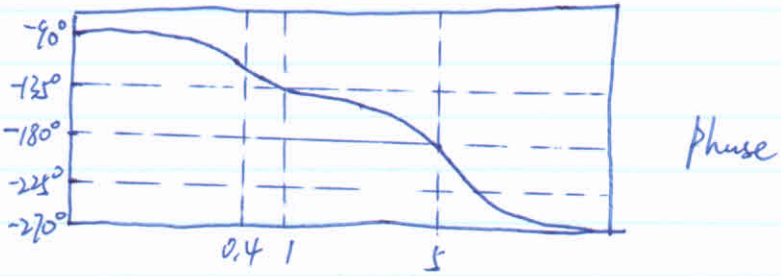
② Very left (lowest frequency) side depends on $\frac{10}{s}$, (or extension which goes through $\omega=1$ ~~lg~~ $L(\omega) = 20\,dB$, slope = $-20\,dB/dec$.

③ Corner frequency, slope increment and phase increment (left to right) can be listed as.

Element	Corner Frequency	Slope Increment	phase Increment
Inertia.	$\omega = \frac{1}{2.5} = 0.4$	-20dB/dec	-90°
1st-order derivative ~~2nd-order derivative~~.	$\omega = 1$	+20dB/dec	$+90^\circ$
2 Oscillation	$\omega = \frac{1}{0.2} = 5$	-40dB/dec	-180°

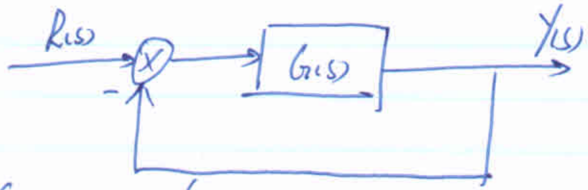
$\zeta = 0.6$. no need to adjust.





§ Frequency Response of Unit negative feedback systems

Block Diagram:



$$\Phi(s) = \frac{G(s)}{1+G(s)} = \frac{Y(s)}{R(s)}$$

§ Stability Analysis of Closed-loop Systems

- In this section, we investigate Nyquist criteria of system stability.

- Using open-loop frequency response to evaluate closed-loop stability.

- Pros: 1) Ploting open-loop Nyquist & Bode plots is easier than ploting closed-loop ones.

  2) ~~These~~ This criterion can < check the closed-loop stability
  check how close the system to unstable (stability margin)

  3) This criterion can provide methods to improve system stability

  4) If system transfer function is unknown, we can obtain the open-loop frequency response from experiments and then check the stability

§ Stability conditions for closed-loop systems

Condition: All poles in the LHS of the complex plane.

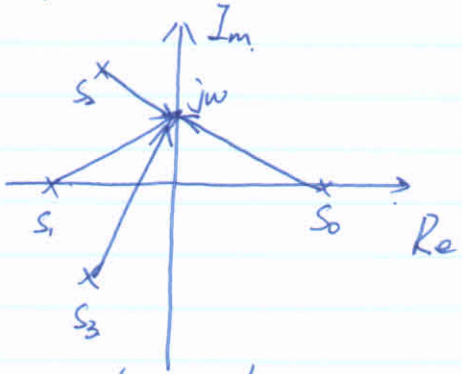
$$D(s) = s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0 = (s-s_1)(s-s_2)\cdots(s-s_n)$$

s_i is the root of characteristic equation $D(s)$

Substitute $s=j\omega$ into $D(s)$, and let $\omega = 0 \to +\infty$.

$$D(j\omega) = (j\omega - s_1)(j\omega - s_2)\cdots(j\omega - s_n)$$

every $j\omega - s_i$ can be described as a vector.



- If S_1 is a real number and in LHs of complex plane, when $\omega = 0 \to +\infty$, phase of $(j\omega - S_1)$ varies from $0° \to 90°$, phase increment is 90°.

- If S_2 & S_3 are conjugate complex poles and in LHS of complex plane. When $\omega = 0 \to +\infty$, sum of phases of vectors $(j\omega - S_2)$ & $(j\omega - S_3)$ ~~phase~~ varies from $0° \to 2\times\frac{\pi}{2}$, phase increment is $2\times\frac{\pi}{2} = \pi$.

- Conclusion: If n ~~roots~~ poles of closed-loop transfer function are all in LHS of complex plane (system is stable) when $\omega = 0 \to +\infty$, increment of $\angle D(j\omega)$ is $n\cdot\frac{\pi}{2}$

$$\boxed{\Delta\angle D(j\omega) = n\cdot\frac{\pi}{2} \quad . \quad \omega = 0 \to +\infty.}$$

- If there exists root in RHS of the complex plane. $\Delta\angle D(j\omega) \neq n\cdot\frac{\pi}{2}$.

§ Relationship between open-loop frequency response & closed-loop stability

Open-loop transfer function

$$G(s)H(s) = \frac{KM(s)}{N(s)}$$

$N(s)$ and $M(s)$ are polynomials of s.

Characteristic equation of closed-loop system

$$D(s) = N(s) + KM(s) = 0$$

Auxiliary function: $F(s) = 1 + G(s)H(s) = \dfrac{N(s)+KM(s)}{N(s)} = \dfrac{D(s)}{N(s)}$

Normally, order of $G(s)H(s)$ is: denominator > numerator, thus den and num of $F(s)$ are with same order "n".

(1) When open-loop system is stable. If so, all roots of $N(s)=0$ are in LHS of complex plane.

$$N(s) = (s-p_1)(s-p_2)\cdots(s-p_n)$$

p_i is open-loop pole of this system, in LHS of complex plane.

Thus $\Delta\angle N(j\omega) = n\cdot\frac{\pi}{2}$, $\omega = 0 \rightarrow +\infty$

$\Rightarrow$ $\Delta\angle F(j\omega) = \Delta\angle D(j\omega) - \Delta\angle N(j\omega)$

Under the assumption: the system is open-loop stable, the above equation should satisfy

$$*.\ \Delta\angle F(j\omega) = \Delta\angle[1+G(j\omega)H(j\omega)] = n\cdot\frac{\pi}{2} - n\cdot\frac{\pi}{2} = 0.$$

Hence, we have.

Conclusion 1: If all open-loop poles are in LHS of the complex plane, the IFF condition for closed-loop ~~stab~~ system to be stable: When $\omega = 0 \to +\infty$

$$\Delta\angle[1 + G(j\omega)H(j\omega)] = 0$$

(2) Open-loop system is not stable.

That means there are P poles in n open-loop poles in LHS of the complex plane, other n-P poles in LHS of the complex plane. In this case we should have

$$\Delta\angle N(j\omega) = (n-P)\frac{\pi}{2} + P(-\frac{\pi}{2})$$

$$= (n-2P)\frac{\pi}{2} \qquad \omega = 0 \to +\infty$$

If closed-loop system is stable

~~of $\Delta\angle F(j\omega) = \Delta$~~

$$\Delta\angle F(j\omega) = \Delta\angle[1 + G(j\omega)H(j\omega)] = \Delta\angle D(j\omega) - \Delta\angle N(j\omega)$$

$$= n\cdot\frac{\pi}{2} - (n-2P)\frac{\pi}{2} = P\pi \qquad \omega = 0 \to +\infty$$

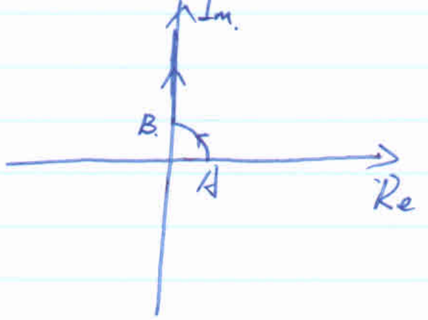
Conclusion 2: In n poles of the open-loop transfer function, if P of them are in RHS of the complex plane, other n-P poles are in LHS of the complex plane then IFF conditions for closed-loop system to be stable is

$$\Delta\angle F(j\omega) = P\pi \quad , \omega = 0 \to +\infty$$

(3) $\frac{1}{s}$ exists in open-loop transfer function

$$G(s)H(s) = \frac{KM(s)}{N(s)} = \frac{KM(s)}{s^{v}N'(s)}$$

$\frac{1}{s}$ corresponds to $(s-0)$ in the denominator. $s=j\omega$, $\omega=0$, vector $(j\omega-0)$ becomes a point, phase is undetermined. To avoid this, adjust the path of $j\omega$ along imag axis a little bit



Adjusted path.

First, starting from $A(\varepsilon, j0)$, along the arc with infinitesimal radius ε to $B(0, j\varepsilon)$:

Second. from $j\varepsilon$ to $j\omega \rightarrow j\infty$.

Because ε is a infinitesimal value, so the second path can be regarded as $\omega = 0_{+} \rightarrow \infty$

If we adjust the path to $A \rightarrow B \rightarrow j\infty$, the pole corresponding to $\frac{1}{s}$ becomes a pole in the LHS of $j\omega$ path. For other poles, it does nothing. Thus,

$$\Delta\angle N(j\omega) = (n-P)\frac{\pi}{2} + P(-\frac{\pi}{2}) = (n-2P)\frac{\pi}{2}$$

$$\Rightarrow \quad \Delta\angle F(j\omega) = \Delta\angle[1+G(j\omega)H(j\omega)] = P\pi. \qquad (0_{+}, j0) \rightarrow (0, j0_{+}) \rightarrow j\infty.$$

Conclusion 3: In n open-loop poles, v of them are at origin $(v \neq 0)$, P of them are ~~at~~ in RHS of complex plane. Iff condition for closed-loop system to be stable is.

when $j\omega$ varies along $(0_+, j0) \rightarrow (0, j0_+) \rightarrow j\infty$.

$$\Delta \angle F(j\omega) = P\pi.$$

§ Nyquist Criteria

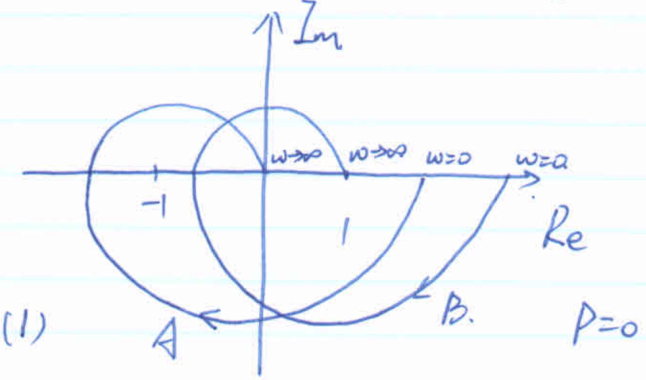
- Use open-loop frequency response to evaluate closed-loop stability

- Conclusion 1 is a special case of Conclusion 2 when $P=0$.

They can be combined as: For open-loop transfer functions with no integral, as $\omega = 0 \rightarrow +\infty$, if $\Delta\angle[1+G(j\omega)H(j\omega)] = P\pi$, then system is stable, otherwise not stable

① If the Nyquist Plot of $G(j\omega)H(j\omega)$ is as shown below

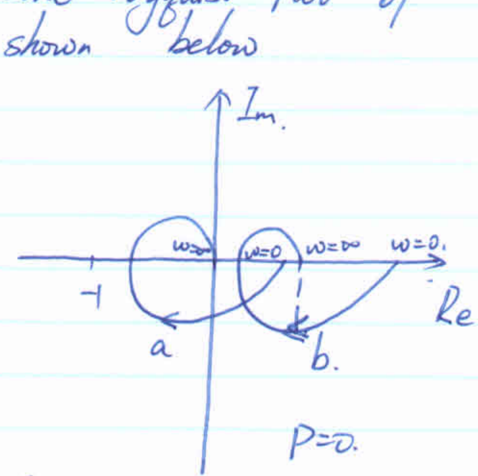
A.



Then $1+G(j\omega)H(j\omega)$ is B. which is A moved to right ~~with~~ by 1.

If the system is open-loop stable, if $\Delta\angle B = 0$ as $\omega=0\rightarrow+\infty$ then system is closed-loop stable. From the above figure we can see that $\Delta\angle B = 0, -\frac{\pi}{2}, -\pi, -\frac{3}{2}\pi, -2\pi$, which is -2π. Obviously, this system is not stable.

② If the Nyquist Plot of $G(j\omega)H(j\omega)$ and $1+G(j\omega)H(j\omega)$ is as shown below



(2)

a is for $G(j\omega)H(j\omega)$; b is for $1+G(j\omega)H(j\omega)$

As $\omega=0\rightarrow\infty$, $\Delta\angle B = 0$, negative angle, 0, positive angle, 0.
$\Delta\angle B=0$. Thus, this system is stable

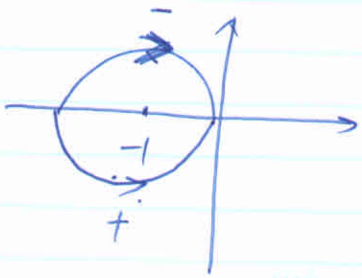
P=0 { In figure (1), curve B encircles origin., closed-loop not stable
(2). curve b doesn't encircle origin, closed-loop stable

· Difference between $G(j\omega)H(j\omega)$ and $1+G(j\omega)H(j\omega)$ is just 1, which means B(b) is a translation of A(a). So we can conclude that

-X-: If $P=0$, $\nu=0$, when the Nyquist plot of ~~system~~ the open-loop system doesn't encircle $(-1, j0)$, this system is closed-loop stable, otherwise, not stable.

The above conclusion can be ~~generalized~~ extended as: When $\nu=0$, $P\neq0$, if the Nyquist Plot of $G(j\omega)H(j\omega)$ bypasses $(-1, j0)$, and the angle increment around $(-1, j0)$ is $P\pi$, then the system is stable, otherwise, not stable.

Note: Increment of angle around $(-1, j0)$



③ Integral exists, i.e. $\nu \neq 0$

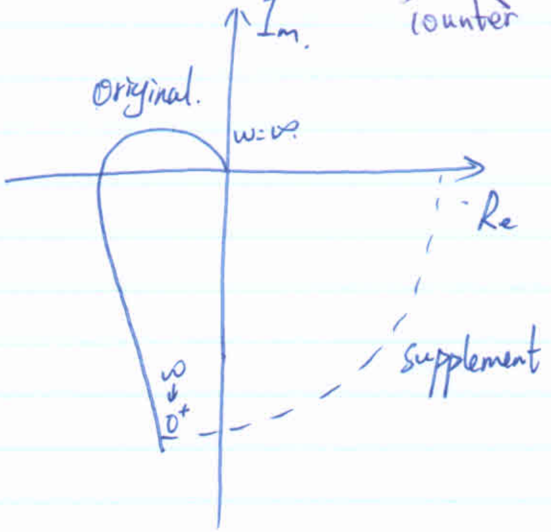
$$G(j\omega)H(j\omega) = \frac{KM(j\omega)}{(j\omega)^{\nu} \cdot N'(j\omega)}$$

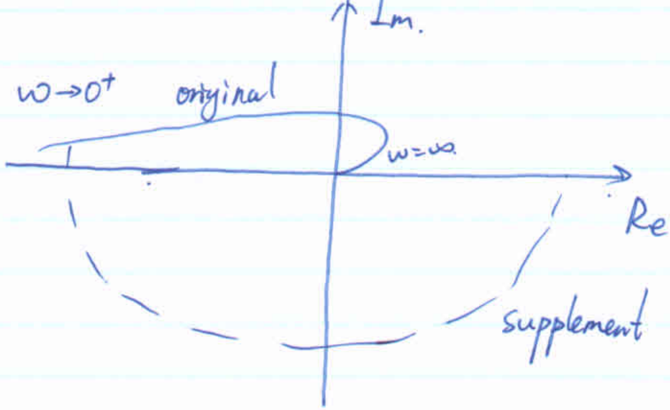
When analyzing stability, the path of $j\omega$ is $(0_+, j0) \to (0, j0_+) \to j\infty$

When $j\omega$ goes along (~~$0_+, j0$~~ → $(0, j0_+) \to j\infty$, the Nyquist plot has already been mentioned. So we just need to add the plot when $\omega = (0_+, j0) \to (0, j0_+)$.

The method is: from $\omega = 0_+$ in the original Nyquist Plot, use ∞ as the radius and counter clockwisely draw $\nu \cdot \frac{\pi}{2}$.

For example:





Supplemented Nyquist plot

⇒ When integral exists, IFF condition for closed-loop system to ($P \neq 0$) be stable is: the supplemented Nyquist Plot of $G(j\omega)H(j\omega)$ bypasses $(-1, j0)$, the angle increment is $P\pi$.

($P=0$) (Supplemented.) Nyquist Plot of $G(j\omega)H(j\omega)$ doesn't encircle $(-1, j0)$

Example: Open-loop transfer function

$$G(s)H(s) = \frac{K(\tau s+1)}{s^2(Ts+1)}$$

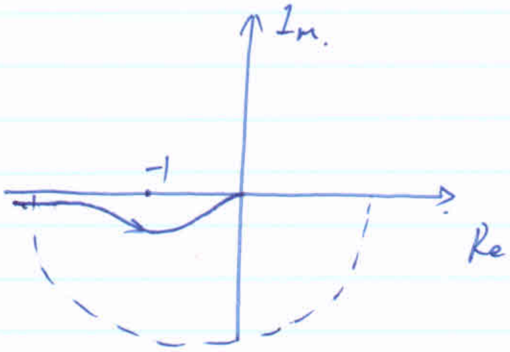
$$K>0, \ T>0, \ \tau>0$$

Use Nyquist Criterion to analyze the stability $T<\tau$, $T>\tau$, $T=\tau$.

Solution: (1) $T<\tau$

$P=0$, $\nu=2$

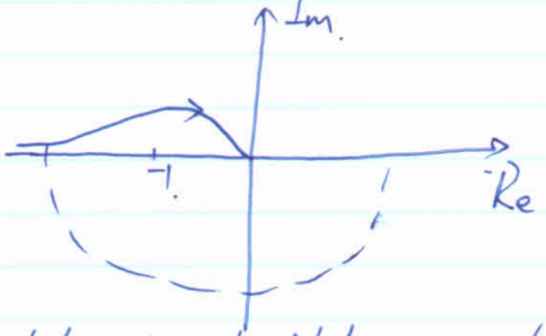
Plot the supplemented Nyquist Plot



The supplemented ~~of~~ Nyquist Plot does not encircle $(-1, j0)$. thus the system is stable

(2) $T > \tau$.

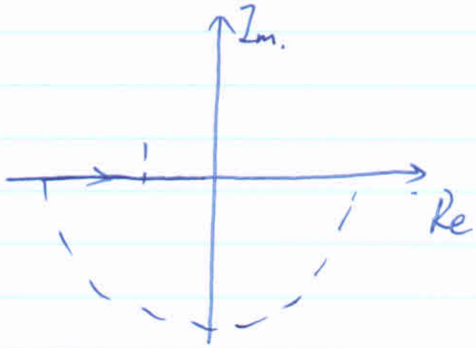
Supplemented Nyquist Plot



Supplemented Nyquist Plot encircles $(-1, j0)$. system is not stable.

(3) $T = \tau$

In this case. open-loop supplemented Nyquist Plot can be shown as



Just goes through $(-1, j0)$, so the closed-loop system is critically/marginally stable.

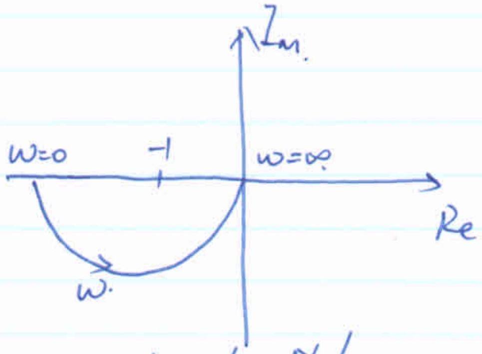
Example: A non-minimum phase system's open-loop transfer function is

$$G(s)H(s) = \frac{K}{Ts-1} \qquad T>0$$

Analyze the stability when $\begin{cases} K>1 \\ 0<K<1 \end{cases}$

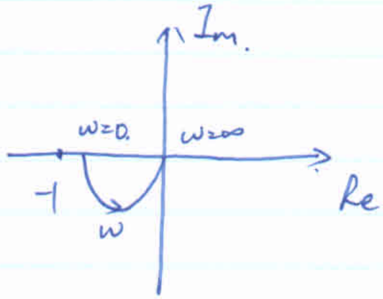
Solution: There is a positive pole. ⇒ $P=1$
No integral $\nu=0$. no need to supplement the Nyquist Plot.

(1) $K>1$



$\omega=0 \rightarrow +\infty$. Nyquist Plot goes around $(-1, j0)$ for $\pi = P\pi$,
⇒ Closed-loop system is stable

(2) $0<K<1$



$\omega=0 \rightarrow +\infty$. Nyquist Plot goes around $(-1, j0)$ for $0° \neq P\pi$
⇒ Closed-loop system is not stable

§ Nyquist Criterion in Bode Plot

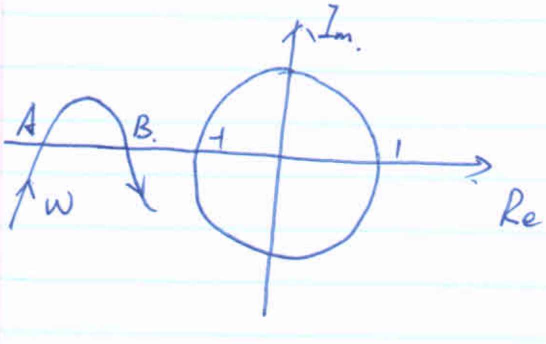
It's difficult to plot Nyquist Plot, while it's easy to ~~plo~~ plot Bode Plot.

Relationship between Nyquist Plot & Bode Plot:

(1) In Nyquist coordinate system, circle with radius = 1, O as origin corresponds to 0dB in $L(\omega)$; Outside unit circle corresponds to the area above 0dB; Inside unit circle corresponds to area below 0dB.

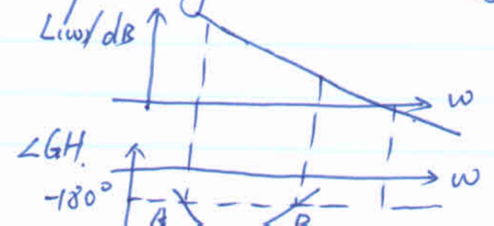
(2) In Nyquist coordinate system, negative real axis corresponds to −180° line in ∠GH in Bode plot.

(3) If open-loop Nyquist Plot clockwisely goes across negative real axis and outside the unit circle, like A, We say



it's "negative crossing" (negative means phase is decreasing). Negative crossing ~~in Bo~~ corresponds to Bode plot: In the area where magnitude response is above 0dB, phase response goes across −180° line towards the decreasing direction.

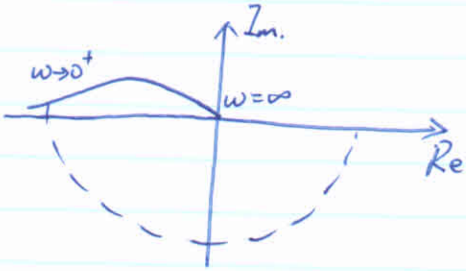
(3) If open-loop Nyquist Plot counter-clockwisely goes across negative real axis and outside the unit circle, like B, We say it's "positive crossing". Positive crossing corresponds to Bode Plot: In the area where magnitude response is above 0dB, phase response goes across −180° towards the increasing direction.

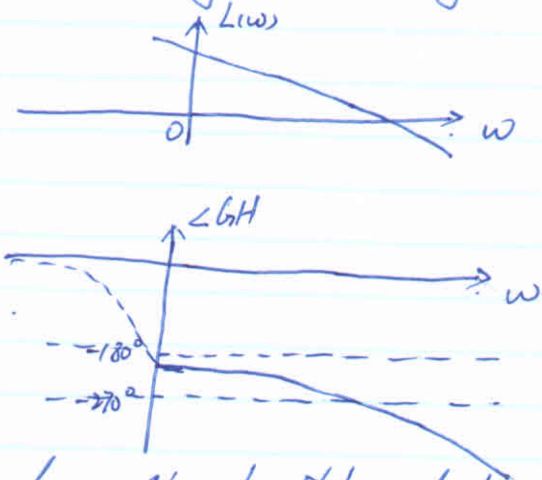


From the above Nyquist Plot we can conclude that:
If open-loop Nyquist Plot (or supplemented plot) does not encircle (-1, j0), then, outside unit circle, number of positive crossing N^{+} and number of negative crossing N^{-} should be equal.

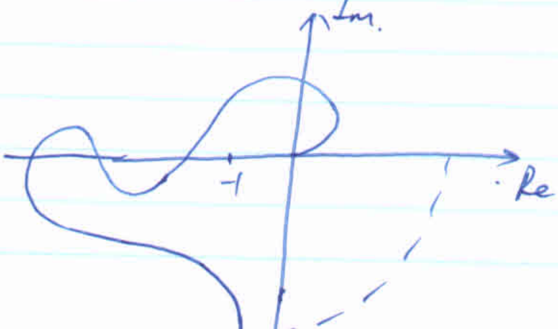
⇒ P=0; If there is no pole of the open-loop transfer function locates in RHS of the complex plane. (P=0). IFF condition for closed-loop system to be stable is: in the open-loop Bode Plot, in the area where magnitude response is above 0dB the phase response (or supplemented) has an equal number of positive and negative crossings ($N^{+}=N^{-}$), or No crossing.

Note that: if there are v integrals in the open-loop transfer function, the method to draw supplemented Bode plot is: at the lowest frequency side in the phase response, extend the curve towards the increasing direction by $v \cdot 90°$





The following plot is an open-loop Nyquist Plot and its supplement



From the above plot we can see that, the Nyquist Plot goes around $(-1, j0)$ for.

$$(N^+ - N^-) \cdot 2\pi$$

For open-loop system with $P>0$, if $(N^+ - N^-) \cdot 2\pi = P\pi$, then closed-loop system is stable; otherwise not stable.

⇒ $P \neq 0$: If the open-loop transfer function has P poles in RHS of the complex plane. IFF condition for closed-loop system to be stable: in open-loop Bode Plot, in the area where magnitude response is above 0dB, the phase (or supplemented) has positive & negative crossing satisfying $N^+ - N^- = P/2$.

Note that, if open-loop Nyquist Plot starts from the negative real axis outside the unit circle, the phase response in Bode Plot will start from $-180°$ line, this is regarded as $\frac{1}{2}$ crossing, positive / negative is determined by the varying trend.

Root Locus

- Stability, dynamics of the closed-loop system depends on the position of the closed-loop poles
- Know the poles → performance
- Put the poles to expected places → expected performance.

However, determining the places of the poles is hard:

1) Closed-loop poles are the roots of the characteristic equations. For 3rd-order / higher order systems, it's really difficult to solve.

2) In system design and tuning, some parameters need to be adjusted oftenly. Usually, it is the Gain. The designer would like to see the changes in poles according to changes in parameters. If solve a new equation with a new set of parameters, that would be tedious

In our design, it can be noticed that:

1) The open-loop poles, zeros are easy to solve. Because the open-loop T.F. consists of some low-order elements in seriel. Numerator & Denominator of open-loop T.F. are the product of some low-order polynomials. Only solving for some low-order equation is enough.

2) It is of significance, if we can directly finds out the moving trend and rules of the closed-loop poles depending only on open-loop poles.

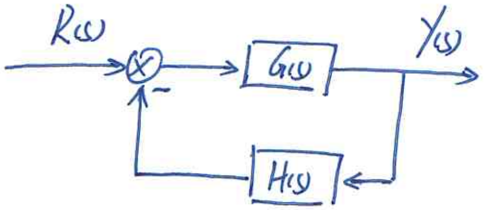
W. R. Evans invented "Root Locus"

Concept of Root Locus:

The locus of the closed-loop pole as one of the system parameters varies from $0 \to \infty$.

In this chapter, we first talk about the case when the open-loop gain varies from $0 \to \infty$, The root locus of this case is called normal root locus.

Negative feedback system



$$\Phi(s) = \frac{G(s)}{1+G(s)H(s)}$$

Characteristic equation $1+G(s)H(s)=0$

$G(s)H(s)=-1$ roots are closed-loop poles

Open-loop transfer function in pole-zero form.

$$G(s)H(s) = \frac{k(s-z_1)(s-z_2)\cdots(s-z_m)}{(s-p_1)(s-p_2)\cdots\cdots(s-p_n)}$$

$z_1, z_2 \cdots z_m$ open-loop zeros.

$p_1, p_2 \cdots p_m$ open-loop poles.

Obviously, as the open-loop gain $0 \to \infty$, k varies from $0 \to \infty$.

$$\sigma + j\omega = \sqrt{\sigma^2+\omega^2}\, e^{j\arctan\frac{\omega}{\sigma}}$$

Characteristic equation

$$\frac{k(s-Z_1)(s-Z_2)\cdots(s-Z_m)}{(s-P_1)(s-P_2)\cdots(s-P_n)} = -1$$

The closed-loop poles must satisfy.

$$|G(s)H(s)| = \frac{k|(s-Z_1)|\,|(s-Z_2)|\cdots|(s-Z_m)|}{|(s-P_1)|\,|(s-P_2)|\cdots|(s-P_n)|} = 1 \qquad (1)$$

$$\begin{aligned}\angle G(s)H(s) = {} & \angle(s-Z_1) + \angle(s-Z_2) + \cdots + \angle(s-Z_m) \\ & -\angle(s-P_1) - \angle(s-P_2) - \cdots - \angle(s-P_n) \\ = {} & \pm(2l+1)\pi \qquad l = 0,1,2\cdots\end{aligned} \qquad (2)$$

(1) & (2) are called root locus equation

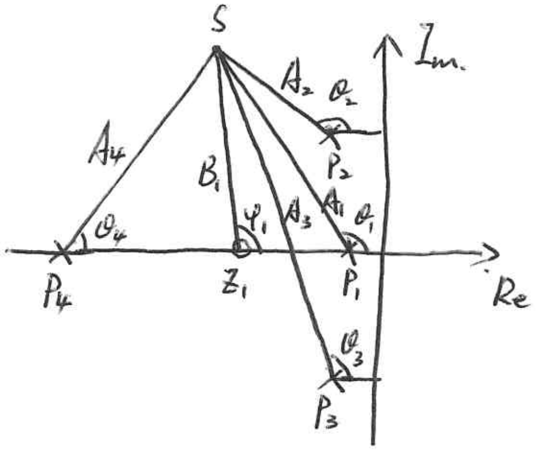
(1) Amplitude condition
(2) Phase condition

All points satisfy (1)&(2) are points on the root locus

From (1). Since k takes the value from $[0,\infty)$, we can always find a k make (1) ~~st~~ hold.

Now, we care if (2) will hold, as long as a point satisfies (2), we can calculate k corresponding to this point, and this point is the closed-loop pole.

If we can find out all points satisfying ~~(1)~~ (2), ~~then~~ ~~poles~~ the set of these poles forms the root locus.



X 4 open-loop poles
O 1 open-loop zero

$$G(s)H(s) = \frac{k(s-Z_1)}{(s-P_1)(s-P_2)(s-P_3)(s-P_4)}$$

Test s.

$\angle G(s)H(s) = \varphi_1 - \theta_1 - \theta_2 - \theta_3 - \theta_4$ if $= (2l+1)\pi$. satisfy phase condition

φ_1 $s-Z_1$
θ_1 $s-P_1$
θ_2 $s-P_2$
θ_3 $s-P_3$
θ_4 $s-P_4$

$$k = \frac{A_1A_2A_3A_4}{B_1}$$

$\Rightarrow$ If test point s satisfy phase condition, then test point s is ~~a~~ the point on the root locus corresponds to $k = \frac{A_1A_2A_3A_4}{B_1}$

OR, s is the closed-loop pole when $k = \frac{A_1A_2A_3A_4}{B_1}$

Steps of drawing Root Locus.

(1) Write open-loop T.F. in pole-zero form.

(2) Find open-loop poles & zeros on. s plane.
O for zero. x for pole.

(3) Find points on s plane satisfying phase condition.
connect them together. → root locus.

(4) For some important points on root locus, use "1"
to calculate k and then open-loop gain. K.

§ Basic rules of drawing root locus.

Negative Feedback characteristic equation

$$G(s)\,H(s) = -1$$

Phase condition

$$\angle G(s)H(s) = \sum_{i=1}^{m} \angle(s-z_i) - \sum_{j=1}^{n} \angle(s-p_j) = (2l+1)\pi \quad l=0,1,2\ldots$$

when k $0 \to +\infty$, the root locus is negative feedback root locus

① ~~B~~ Number of branchs of root locus

According to the definition. root locus. is the variation of closed-loop poles on s-plane as one parameter changes.

Thus, ~~not~~ number of branchs is the same as number of ~~roots~~ poles. For open-loop poles n > m. zeros, closed-loop characteristic equation n-th order algerabic equation. If m>n. , m branchs.

Number of branchs = max[m, n]

② Continuity & Symmetry.

1) Continiuous.

2) Roots : Real / Conjugate , symmetric according to real axis.

③ start & End point.

start point: s positions when $k=0$.
end point: s positions when $k=\infty$

closed-loop characteristic equation:

$$(s-p_1)(s-p_2)\cdots(s-p_n)+k(s-z_1)(s-z_2)\cdots(s-z_m)=0$$

$k=0$. $(s-p_1)(s-p_2)\cdots(s-p_n)=0$. roots = open-loop poles

$$\frac{1}{k}(s-p_1)(s-p_2)\cdots(s-p_n)+(s-z_1)(s-z_2)\cdots(s-z_m)=0$$

$k=\infty$ $(s-z_1)(s-z_2)\cdots(s-z_m)=0$ roots = open-loop zeros

Root locus starts from open-loop poles, end to open-loop zeros.
$n>m$. n starts m end, <u>the else $n-m$?</u>

④ Asymptote

We should find the other $n-m$ root locus

From the phase condition. in infinity place of the s-plane, there exist points satisfying phase condition

Infinity place $|s|\to\infty$, we can consider all those the vectors of ~~points~~ poles, ~~to~~ & zeros to s. have the same ~~phase~~. argument. $=\Phi$

phase condition: $m\Phi-n\Phi=\pm(2l+1)\pi$ $l=0,1,2,\ldots$

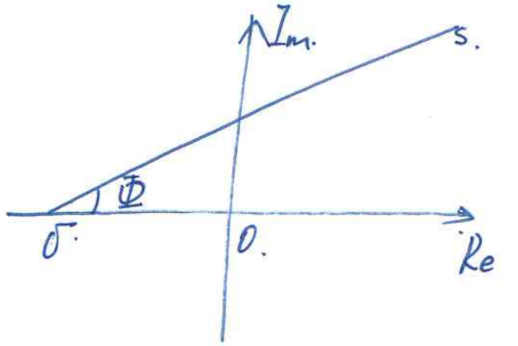
The ray: starting from poles/zeros ending at s. The arg of this ray is the asymptote arg.

far away side is root locus (root locus asymptote)

Arg of the asymptote: $\boxed{\Phi = \frac{(2l+1)\pi}{n-m} \quad l=0,1,2,\cdots}$

Root locus is ~~asym~~ symmetric to real axis, and according to $\Phi = \frac{(2l+1)\pi}{n-m}$, we can have $(n-m)$ asymptotes.

Root locus is symmetric to real axis, thus, these root locus must intersect at real axis σ.



Open-loop T.F.

$$G(s)H(s) = k\cdot\frac{s^m + \sum_{i=1}^{m}(-z_i)s^{m-1}+\cdots+\prod_{i=1}^{m}(-z_i)}{s^n+\sum_{j=1}^{n}(-p_j)s^{n-1}+\cdots+\prod_{j=1}^{n}(-p_j)}$$

$$= \frac{k}{s^{n-m}-\left(\sum_{j=1}^{n}p_j-\sum_{i=1}^{m}z_i\right)s^{n-m-1}+\cdots}$$

Because $k\to\infty$, $s\to\infty$. Denominator can only keep first two terms

$$G(s)H(s) = \frac{k}{s^{n-m}-\left(\sum_{j=1}^{n}p_j-\sum_{i=1}^{m}z_i\right)s^{n-m-1}} = -1$$

$$\Rightarrow \left| s^{n-m} - \left(\sum_{j=1}^{n} p_j - \sum_{j=1}^{m} z_i\right) s^{n-m-1} \right| = k.$$

Infinity s.: distance to each pole/zero is the same. equal to <u>s to σ</u>.

$$\left| (s-\sigma)^{n-m} \right| = k.$$

Newton's binomial Theorem:

$$\left| s^{n-m} - (n-m)\sigma s^{n-m-1} \right| = k.$$

$$(n-m)\sigma = \sum_{j=1}^{n} p_j - \sum_{j=1}^{m} z_i$$

$$\boxed{\sigma = \frac{\sum_{j=1}^{n} p_j - \sum_{j=1}^{m} z_i}{n-m}}$$ $n>m$.

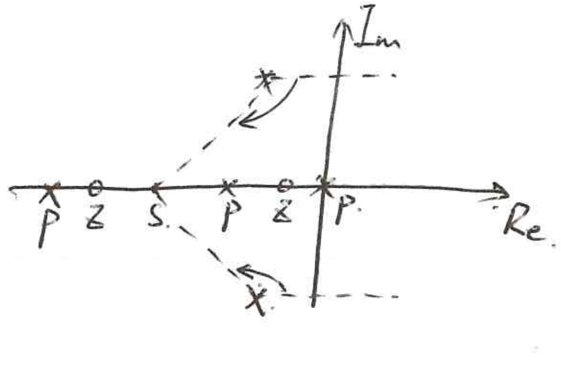
⑤ Root locus on real axis

Take a point on real axis as test point to verify if it satisfies phase condition.

Open loop zeros/poles → real, conjugate complex numbers

Conjugate poles point to "s", Σarg = 0.

pole/zeros on the LHS of "s" arg = 0.
pole/zeros on the RHS of "s" arg = π



Thus, when "s" has odd number of pole & zeros on is RHS, "s" is on the root locus.

⇒ Segments on the real axis with odd number of pole & zeros on their RHS belong to root locus.

Example1: Negative Feedback System open-loop T.F.

$$G(s)\,H(s) = \frac{k}{s(s+1)(s+2)}$$

Draw a rough Root Locus.

Solution: 1) Three real open-loop poles. $p_1=0$ $p_2=-1$, $p_3=-2$.

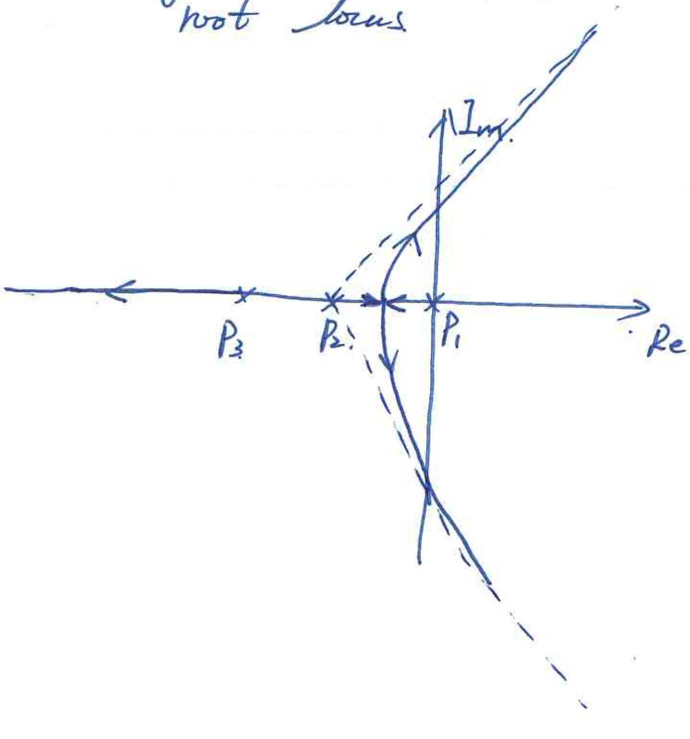
2) Three root locus. starting from p_1, p_2, p_3, respectively.

3) Three asymptote

$$\Phi = \frac{(2k+1)\pi}{3-0} = \frac{\pi}{3},\ \frac{2\pi}{3},\ \pi$$

$$\sigma = \frac{-3}{3-0} = -1.$$

4) Segment between p_1, p_2 and LHS of p_3 (ray) are root locus



Examples: NF OL. TF

$$G(s)H(s) = \frac{k(s+2)}{s(s+1)}$$

Solution: $S_1 = 0$ $S_2 = -1$

$Z_1 = -2$.

1) Two root locus. starting from P_1, P_2 respectively.

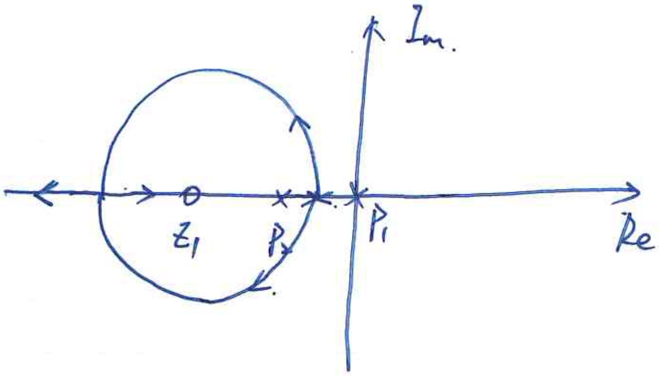
2) One root locus ends at Z_1

The other one approaches asymptote

$$\Phi = \frac{(2k+1)\pi}{2-1} = \pi$$

$$\sigma = \frac{-1+2}{2-1} = 1. \quad \text{does not matter}$$

3) Segment between P_1, P_2, LHS of Z_1 root locus



⑥ Rendezvous & Separation point

From the above two examples we notice that, two or more root locus may rendezvous to a point and then separate. It's called rendezvous & separation point.

Open-loop T.F: $G(s)H(s) = \frac{kM(s)}{N(s)}$

Characteristic equation:

$$D(s) = N(s) + kM(s)$$

Use RS point to denote rendezvous & separation point. and it is ~~a~~ α. That means, when k equals a specific value two or more root locus rendezvous & separate at α. D(s) = 0 has multiple roots at α.

$$D(s) = 1 + G(s)H(s) = 0$$

D(s) has $(s-\alpha)^2$ or higher order. Thus, in $\frac{dD(s)}{ds}$, there must exists $(s-\alpha)$. Hence, α must be the root of

$$\frac{dD(s)}{ds} = 0.$$

$$\frac{dD(s)}{ds} = \frac{d}{ds}[1 + G(s)H(s)] = 0$$

$$\frac{d[G(s)H(s)]}{ds} = \frac{d}{ds}\left[\frac{kM(s)}{N(s)}\right] = 0.$$

$$\frac{d}{ds}\left[\frac{kM(s)}{N(s)}\right] = 0 \qquad \text{or} \qquad \frac{d}{ds}\left[\frac{N(s)}{M(s)}\right] = 0$$

$\Rightarrow$ $M'(s)N(s) - N'(s)M(s) = 0$ or $N'(s)M(s) - M'(s)N(s) = 0$ ①
root~~s~~ is α.

① is ~~a~~ appropriate for m=0, i.e., no open-loop zero.

If m=0. $\frac{d}{ds}[N(s)] = 0.$

Note: (1) α is root of ①. But not all the roots of ① are RD point, We should choose by fool following other rules.

(2) α can be complex point (conjugate).

Another rule (without proof): when l root locus enter and leave RD point, two neighbor root locus.'s angle is

$$\frac{2i+1}{l}\pi \quad i=0.1.2,\dots$$

- angle: tangent of those root locus. at α.
- $l=2$ $\frac{\pi}{2}$.

Example 3: Solve for RD point in Example 1.

Solution: No open-loop zero.

$$\frac{d}{ds}[s(s+1)(s+2)]=0$$

$$3s^2+6s+2=0.$$

$$s_1=-0.423 \qquad s_2=-1.577$$

Because -1.577 is not on root locus Only -0.423 is valid for RD point.

Example 4: Solve for RD points A.B. in Example 2.

Solution: $M(s)=s+2$, $N(s)=s(s+1)$.

$$\frac{d}{ds}\left[\frac{s+2}{s^2+s}\right]=0$$

$$s^2+4s+2=0$$

$s_1 = -3,41$ $s_2 = -0,59$.

s_1, s_2 are both on root locus. A $-0,59$ B $-3,14$

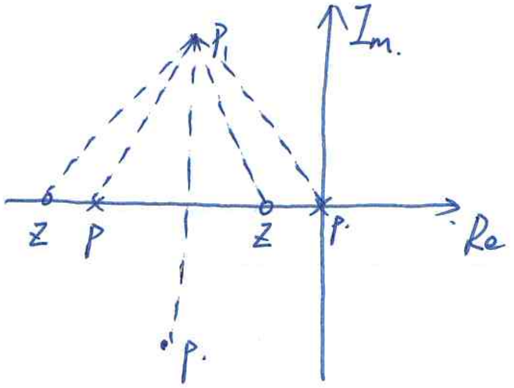
⑦ Angle of incidence and departure

The root locus starting from the complex open-loop pole; the angle between the tangent at starting point and ~~to~~ positive real axis is Angle of ~~incidence~~. departure.

The root locus ending at the complex open-loop zeros: the angle between the tangent at ending point and positive real axis is Angle of incidence.

Suppose, open-loop T.F has m zeros
n poles

p_1 is a complex pole. Take p_1 as an example to calculate Angle of departure. θ_{p_1}



Firstly, find $\bar{s}$ near p_1 satisfying phase condition.

$$\sum_{i=1}^{m} \angle(\bar{s}-z_i) - \angle(\bar{s}-p_1) - \sum_{j=2}^{n} \angle(\bar{s}-p_j) = \pm(2l+1)\pi. \quad l=0,1,2\ldots$$

Because $\bar{s}$ is near p_1, so $\angle(\bar{s}-z_i)$, $\angle(\bar{s}-p_j)$ can be replaced by $\angle(p_1-z_i)$, $\angle(p_1-p_j)$, respectively.

$\angle(\bar{s}-p_1)$ is the Angle of departure θ_{p_1}.

Thus. $\theta_{p_1} = \mp(2l+1)\pi + \sum_{j=1}^{m}\angle(p_1 - z_i) - \sum_{j=2}^{n}\angle(p_1 - p_j) \quad l = 0, 1, 2 \cdots$

Similarly, θ_{z_1} (Angle of incidence at complex open-loop zero z_1):

$$\theta_{z_1} = \pm(2l+1)\pi + \sum_{j=1}^{n}\angle(z_1 - p_j) - \sum_{i=2}^{m}\angle(z_1 - z_i) \quad l = 0, 1, 2 \cdots$$

Easy to get the symmetric Angles for conjugate poles and zeros.

Example 54: Negative feedback system open-loop T.F.

$$G(s)H(s) = \frac{k(s+1)}{s^2 + 3s + 3.25}.$$

Try to draw the root locus.

Solution: Two open-loop poles.

$p_1 = -1.5 + j$

$p_2 = -1.5 - j$

One open-loop zero.

$z_1 = -1$

Two root locus, starting from p_1, p_2. one ends at z_1 the other one goes to $-\infty$ on real axis

$$\Phi = \frac{(2l+1)\pi}{n-m} = \pi.$$

RD point can be solved by.

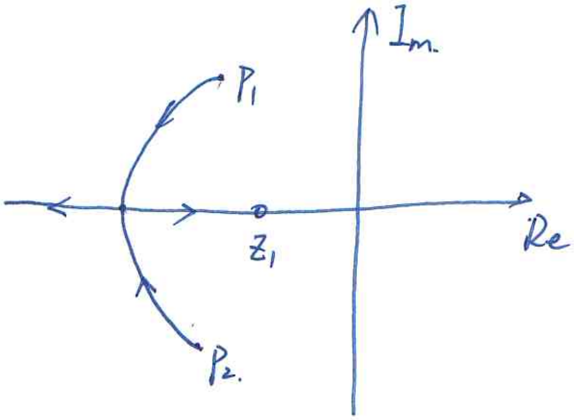
$$\frac{d}{ds}\left[\frac{s^2+3s+3.25}{s+1}\right] = 0$$

$$s^2 + 2s - 0.25 = 0$$

$S_1 = -2.12$ $\qquad$ $S_2 = 0.12$.

S_2 does not satisfy RD point . so RD point -2.12.

$$\theta_{P_1} = \pi + \angle(P_1 - Z_1) - \angle(P_1 - P_2) = 180° + 116.6° - 90° = 206.6°$$



⑧ Point of intersection with Image axis.

When root locus intersects with the imag axis, that means , k equals to a specific value, characteristic equation has pure imaginary roots $\pm j\omega$.

$S = j\omega. \rightarrow 1 + G(s)H(s) = 0.$

$1 + G(j\omega)H(j\omega) = 0$

Re & Im $= 0$

$Re[1 + G(j\omega)H(j\omega)] = 0.$

$Im[1 + G(j\omega)H(j\omega)] = 0.$

Two equations, two unknowns, solve for k & ω.

k : critically stable k.

jω: ~~is~~ point of intersection on Im.

k can also be calculated from other methods, like Routh Rules.

Example: Unit negative feedback open-loop T.F.

$$G(s)H(s) = \frac{k}{s(s+1)(s+2)}.$$

Calculate k and point of intersection on Im.

Method 1.

Solution: Characteristic equation:

$$D(s) = 1 + \frac{k}{s(s+1)(s+2)} = 0.$$

$s = j\omega$.

$$-j\omega^3 - 3\omega^2 + 2j\omega + k = 0$$

$$\begin{aligned} Re: &\ -3\omega^2 + k = 0. \\ Im: &\ -\omega^3 + 2\omega = 0 \end{aligned} \Bigg\} \Rightarrow \begin{cases} \omega = 0 \\ k = 0 \end{cases} \quad \begin{cases} \omega = \pm\sqrt{2}. \\ k = 6. \end{cases}$$

$k=0$: closed-loop pole is at 0.

$k=6$: critically stable, closed-loop poles are at $\pm j\sqrt{2}$.

Method 2:

$$D(s) = s^3 + 3s^2 + 2s + k = 0.$$

Routh's rules

$$\begin{array}{lll} s^3 & 1 & 2. \\ s^2 & 3 & k. \\ s^1 & \frac{6-k}{3} & 0 \\ s^0 & k. & \end{array}$$

Two cases: (1) $k=6$. s^1 row 0.

auxiliary equation: $3s^2 + k = 0$.

$s = \pm j\sqrt{2}$.

(2) $k=0$. s^0 row 0.

$s=0$.

⑨ Calculation of open-loop gain k. K.

After finishing a draft of root locus, according to the phase conditions. We need to indicate k for some important points on root locus.

For a point $\bar{s}$ on root locus, the corresponding $\bar{k}$ can be calculated by

$$\bar{k} = \frac{|\bar{s}-p_1||\bar{s}-p_2|\cdots|\bar{s}-p_n|}{|\bar{s}-z_1||\bar{s}-z_2|\cdots|\bar{s}-z_m|}$$

open-loop gain

$$K = k\frac{\prod_{i=1}^{m}(-z_i)}{\prod_{\underset{\downarrow}{\underline{j=v}}}^{n}(-p_j)}$$

Remove 0 pole here.

⑩ Sum and Product of closed-loop poles

When some of the poles have been determined, we can use the relationship between the roots & ~~of~~ coefficients of the characteristic equation to locate other poles

Characteristic equation roots: $s_1, s_2 \cdots s_n$.

$$D(s) = s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0 = (s-s_1)(s-s_2)\cdots(s-s_n)$$

$$\sum_{i=1}^{n} s_i = -a_{n-1}$$

$$\prod_{i=1}^{n}(-s_i) = a_0.$$

Example: For $G(s)H(s) = \frac{k}{s(s+1)(s+2)}$. the root locus has been drafted beforeed.

(1). Intersection point at Im. $s_{1,2} = \pm j\sqrt{2}$.
please determine the 3rd closed-loop pole

(2) Open-loop gain K when critically stable

Solution: (1) Characteristic equation
$D(s) = s^3 + 3s^2 + 2s + k = 0$

$s_1 + s_2 + s_3 = -3$
$s_3 = -3 - s_1 - s_2 = -3$

(2) $k = 6$.

$$K = k \frac{\prod_{i=1}^{m}(-z_i)}{\prod_{j=2}^{n}(-p_j)} = 6 \times \frac{1}{1 \times 2} = 3$$

Example: Open-loop T.F.

$$G(s)H(s) = \frac{k}{s(0.366s+1)(0.5s^2+s+1)}$$

Draw a draft of the root locus.

Solution: Write the open-loop T.F. in a pole-zero format

$$G(s)H(s) = \frac{\bar{k}}{s(s+2.73)(s^2+2s+2)}$$

No open-loop zero $m = 0$.
Four open-loop poles $p_1 = 0$ $p_2 = -1+j$ $p_3 = -1-j$ $p_4 = -2.73$.
$n = 4$.

Rule ①: $n=4$. 4 root locus

Rule ②: 4 root locus are continuous, symmetric to real axis

Rule ③: Starting from $p_1, p_2, p_3. p_4$.

Rule ④: $n-m=4$. ⇒ 4 root locus approach asymptote, respectively

$$\sigma = \frac{\sum_{j=1}^{n} p_j - \sum_{i=1}^{m} z_i}{n-m.}$$

$$= \frac{0 -2.73-1+j-1-j}{4} = -1.18$$

$$\Phi = \frac{(2l+1)\pi}{n-m} = \pm\frac{\pi}{4}, \pm\frac{3\pi}{4},$$

Rule ⑤: On real axis $[0, -2.73]$ is root locus.

Rule ⑥: Root locus starting from $p_1=0$ $p_4=-2.73$. will rendezvous & separate

$$\frac{d}{ds}[s(s+2.73)(s^2+2s+2)] = 0.$$

The root of the above equation is -2.06, which is in the range $[0, -2.73]$. Thus, two root locus will rendezvous & separate at -2.06.

Rule ⑦: For root locus starting from $p_{2.3} = -1\pm j$. angle of departure can be calculated by.

$$\theta_{p_2} = \mp(2l+1)\pi - \angle(p_2-p_1) - \angle(p_2-p_3) - \angle(p_2-p_4)$$

$$= \mp(2l+1)\pi - 45^\circ + 90^\circ - 30^\circ$$

$$= \mp 75^\circ$$

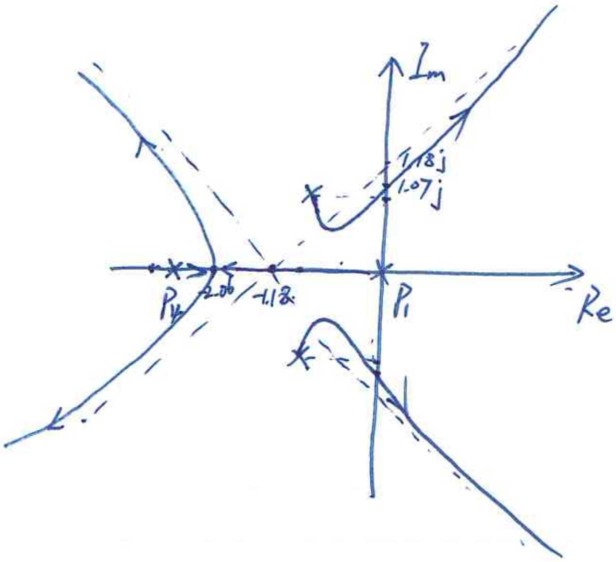
Rule ⑧: Intersections with Im.

$$\begin{cases} \omega^4 - 7.46\omega^2 + \bar{k} = 0. \\ -4.73\omega^3 + 5.46\omega = 0 \end{cases}$$

$\bar{k} = 0$ $\quad$ $\omega = 0.$

$\bar{k} = 7.1$ $\quad$ $\omega = \pm 1.07$. Say. root locus starting from $P_{2,3}$ will interact Im at $\pm j1.07$. corresponding $\bar{k} = 7.1$

Rule ⑨: $K = 1.33$



Example: Open-loop T.F.

$$G(s)H(s) = \frac{k}{s(s+4)(s^2+4s+20)}$$

Draw the root locus.

Solution: $p_1 = 0 \quad p_2 = -4 \quad p_{3,4} = -2 \pm j4$

$n = 4 \quad m = 0$

Rule ④: 4 root locus. $\sigma = -2$. Φ: $\pm 45° \quad \pm 135°$

Rule ⑤: Real axis $[-4, 0]$ is root locus.

Rule ⑥: $\frac{d}{ds}[s(s+4)(s^2+4s+20)] = 0$

$s_1 = -2. \qquad s_{2,3} = -2 \pm j\,2.45.$

$s_1 = -2$ is the R&D point on real axis.

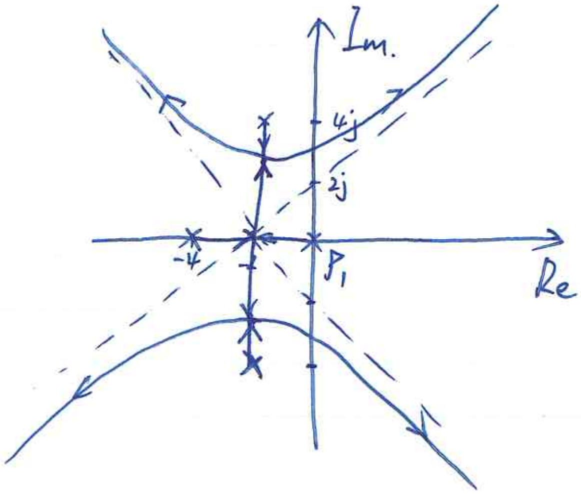
Rule ⑦: $\theta_{p_3} \quad \theta_{p_4} = \mp 90°$ Thus. $-2 \pm j\,2.45$ are R&D points

Rule ⑧:

$$\begin{cases} \omega^4 - 36\omega^2 + k = 0. \\ -8\omega^3 + 80\omega = 0 \end{cases}$$

$$\begin{cases} \omega = 0 \\ k = 0 \end{cases} \qquad \begin{cases} \omega = \pm\sqrt{10} \\ k = 260 \end{cases}$$

Rule ⑨: $K = 3.25$



Example: Open-loop T.F.

$$G(s)\,H(s) = \frac{k(s+0.4)}{s^2(s+3.6)}$$

Draw the root locus.

Solution: $p_1=0$ $p_2=0$ $p_3=-3.6.$ $z_1=-0.4$

3 root locus. one of them ends at z_1

Rule ④: $\Phi = \pm 90^\circ$ $\sigma=-1.6$

Rule ⑤: Real axis $[-0.4, -3.6]$ is root locus.

Rule ⑥: $s^3+2.4s^2+1.44s=0.$

R&D points: $s_1=0$ $s_2=-1.2..$ At $s_2=-1.2$ $k=4.32.$

Rule ⑥ proves that. for $M(s)N'(s)-M'(s)N(s)=0.$
the roots must have 2 identical roots.
Similarly. if characteristic equation has 3 identical roots. $(D(s)=0)$. then $\frac{d^2}{ds^2}[D(s)=0]$ must has

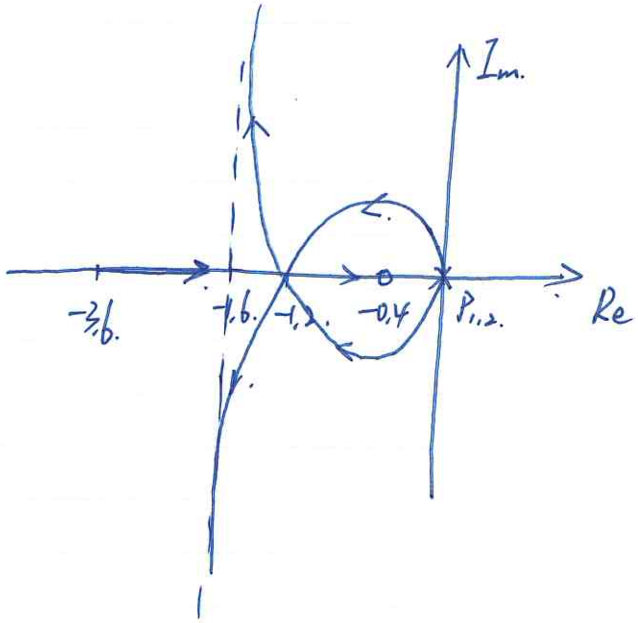
$D(s)=0$'s 3 identical roots

$$\frac{d^2}{ds^2}[D(s)=0] = \frac{d^2}{ds^2}[s^2(s+3.6)+4.32(s+0.4)]$$

$$=6s+7.2=0. \qquad \underline{s=-1.2}$$

Thus, the system has 3 identical poles (closed-loop at -1.2..
Three root locus will rendezvous & separate at -1.2.

According to the auxiliary rule of Rule ⑥. angles between 3 root locus at -1.2 is 60°.



§ Root locus & Stability

Why use Root locus.

We can analyze the system stability from. root locus.

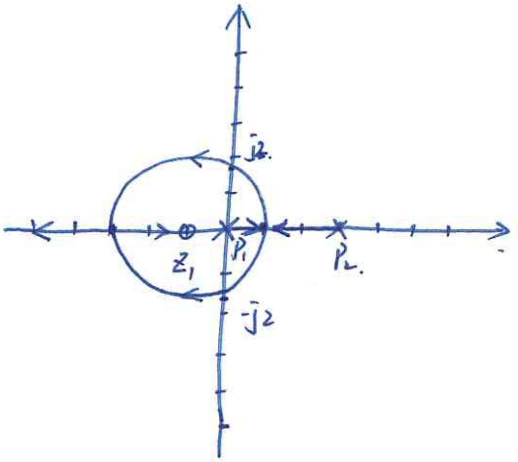
Example: Open-loop transfer function

$$G(s)H(s) = \frac{k(s+1)}{s(s-3)}$$

use root locus to evaluate the system stability

Solution: $P_1 = 0.$ $P_2 = 3.$ $n = 2.$

$Z_1 = -1$ $m = 1$



R& D points on real axis +1, -3.

~~R~~ Intersection points on imag axis $\pm j\sqrt{3} \rightarrow \begin{cases} k=3 \\ k=1 \end{cases}$

(1) Stability analysis: For the closeed-loop system. when. open-loop Gain $K < 1$. the closed loop system is not stable.

(2) Dynamic process analysis: At one of the R&D point $(-3,0)$.

① $k = 9$. $K = 3$. That means that when. $1 < K < 3$. closed-loop T.F has a pair of conjugate complex poles. and a zero at -1. the system is stable, but will include decaying oscillation in the response.

② $k > 9$, $K > 3$., two negative real poles and one zero at $z_1 = -1$. system is stable, no decaying oscillation.

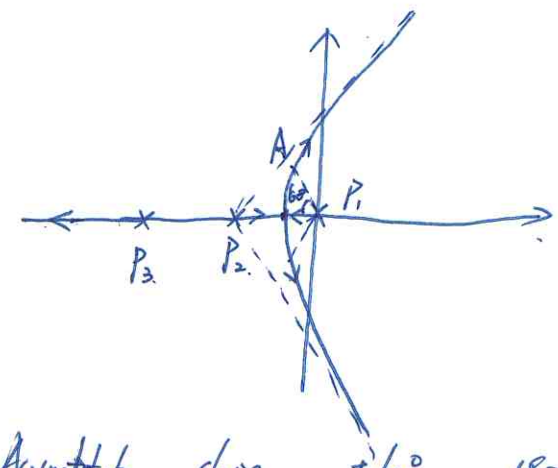
(3) Steady state error analysis: open-loop TF has $p_1 = 0$. so the system is Type I system According to the positions of closed-loop poles. we can calulate k & K. $K = K_v$. then e_{ss}.

Example: Negative feedback system:

$$G(s)H(s) = \frac{k}{s(s+1)(s+2)}$$

Use root locus to analyze the system performance. Calculate performance indities when $\zeta = 0.5$.

Solution: $p_1=0$ $p_2=-1$ $p_3=-2$. $n=3$ $m=0$.



Asymptote slope. $\pm 60°$ $180°$ $\sigma=-1$

R&D $\alpha=-0.423$ $k=0.385$

(1) stability analysis: Intersection points on imag axis $\pm j\sqrt{2}$.
$k=6$. $K=3$ So $0<K<3$. system stable

(2) Dynamic process analysis: $\alpha=-0.423$. R&D $k=0.385$ $K=0.163$

① when $k<0.385$ $K<0.163$. three negative real poles. Step response. monotonically increasing.

② $0.163<K<3$. one pair of conjugate complex poles + one negative real pole. (left to -2) Satisfy closed-loop dominant poles. The system will show the property of under-damped 2nd-order system. As k increases, closed-loop complex poles move along root locus $\sigma_p\uparrow$. $t_s\uparrow$, $t_p\downarrow$. $\zeta\omega_n\downarrow$. $\omega_n\sqrt{1-\zeta^2}\uparrow$

(3) $\zeta=0.5$. $-\zeta\omega_n\pm\omega_n\sqrt{1-\zeta^2}$ $\frac{\sqrt{1-\zeta^2}}{\zeta}=\sqrt{3}$. $60°$

A: $-0.334+j0.571$ $k=1.06$ $K=0.503$

$\sigma_p=16\%$ $t_s=24s$ Type I. $K_v=K=0.503$

(4) steady state error analysis: Type I system. $K_v = K = 0.503$.
Step signal : $e_{ss} = 0$
Ramp signal: $e_{ss} = 1/K_v$.
Acc signal : $e_{ss} = \infty$.